FIXED POINT FORMULAS AND LOOP GROUP ACTIONS

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Dedicated to the memory of Professor Fredrick Almgren, Jr.

1. Introduction

In this paper we present a new fixed point formula associated with loop group actions on infinite dimensional manifolds. This formula provides information for certain infinite dimensional situations similarly as the well known Atiyah-Bott-Segal-Singer's formula does in finite dimension. A generalization of the latter to orbifolds will be used as an intermediate step.

There exist extensive literature on loop groups, loop algebras and their representations. What set the present work and [C1] apart is the focus on representations of the central extensions of the loop groups, induced from Hamiltonian actions; getting explicit formulas which determine the multiplicities of the irreducible highest weight components, hence the structure of the induced represenations.

The results in [C1] show that the highest weight vector occurring in an induced representation are carried by a compact variety, provided the setup meets certain general conditions. In particular geometric quantization is generalized to the setting of loop group actions. The results here link the representations with local data on fixed points. In that direction, we also obtain a new multiplicity formula.

The current paper can be read either as a sequel to [C1] or on its own.

The original motivation was to understand in geometric setting a conjecture by Verlinde [V]. From it the better known Verlinde formula was derived [V, MS]. As the project progressed, it became clear that Verlinde's conjecture is the tip of an iceberg. Representations of the central extensions of loop groups, induced from Hamiltonian actions on infinite dimensional manifolds, can be related to local geometry at the fixed point sets.

Application in a forthcoming paper will include a direct proof of the aforementioned conjecture.

More interestingly the we construct a class of G-orbifolds called fusion product which are geometric dual to the product in Verlinde fusion algebra.

Much has been done about Verlinde formula, we refer the readers to [F,Be] for references.

1.1. **Some notations.** Let G be a connected and simply connected compact simple Lie group, and T be a maximal torus. Let W be its Weyl group, $\mathfrak{t}_+$ be the positive Weyl chamber and P, P_+ the sets of weights and dominant weights respectively, after fixing a set of simple roots. Let $D = \prod_{\alpha>0}(1-e^{-\alpha})$ denote the Weyl denominator, θ be the highest root of $\mathfrak{g}$. On $\mathfrak{g}$, fix an invariant bilinear form $(\cdot|\cdot)$, so that $(\theta^{\mathrm{v}}|\theta^{\mathrm{v}}) = 2$ where θ^{v} is the coroot corresponding to θ. The bilinear form induces a

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map $\nu : \mathfrak{t} \to \mathfrak{t}^*$ and a bilinear form on $\mathfrak{t}^*$. The lattice in $\mathfrak{t}$ generated by $\{W(\theta^\vee)\}$ is denoted by M, and

$$M^* = \{t \in \mathfrak{t} \big| \, (t|n) \in \mathbb{Z}, \forall n \in M\}.$$

Let $h^\vee$ be the dual Coxeter number, ρ the half sum of positive roots. Define the affine alcove and the set of dominant weights in there respectively as

$$C = \mathfrak{t}_+ \cap \{a| \, (a|\theta) \leq 1\}, \quad P_+^k = \{\lambda \in P_+| \, (\lambda|\theta) \leq k\} = P_+ \cap kC.$$

The face C^{aff} in C defined by $(\cdot|\theta) = 1$ is special, in particular

$$\partial(\cup_{w \in W} wC) = \cup_{w \in W} wC^{\text{aff}}.$$

The lattice $\frac{M^*}{k+h^\vee}$ induces a finite subgroup of T, $\exp(2\pi\nu^{-1}\frac{M^*}{k+h^\vee})$. The subset

$$\{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\} \tag{1.1}$$

plays an important role in this paper. An useful observation is that each element τ in that subset is a regular element of G.

1.2. **Assumptions on X.** Let X be a Banach manifold on which LG acts, ω be an invariant symplectic form and μ be the associated moment map at level $k \in \mathbb{Z}_+$, i.e.,

$$\mu : X \to l\mathfrak{g}^0 \times \{k\}$$

where elements of $l\mathfrak{g}^0$ have one degree less of differentiability than those in $l\mathfrak{g}$. Being a moment map, μ is equivariant with respect to the LG-action on X and the co-adjoint action on $l\mathfrak{g}^0 \times \{k\}$. The co-adjoint action loses one degree of differentiability, hence the target of μ is $l\mathfrak{g}^0 \times \{k\}$. We assume in this paper the following unless stated otherwise:

H1: μ proper;
H2: $\mu(X)$ is transversal to $\mathfrak{t} \times \{k\}$. More details are given in Section 2.

1.3. **The highest weight modules of the induced representations.** In [C1], we studied holomorphic actions by a loop group LG, when X is complex. Let $N^+ \subset LG^{\mathbb{C}}$ denote the subgroup whose Lie algebra is $sum_{\alpha>0}(l\mathfrak{g})^{\mathbb{C}}_\alpha$. It consists of non-constant boundary values of holomorphic maps from the unit disk to $G^{\mathbb{C}}$, together with constant maps with values in the positve nilpotent group of $G^{\mathbb{C}}$.

Suppose L is an holomorphic LG-line bundle over X. We proved in [C1] that there is a compact complex T-orbifold X_N and an orbifold T-line bundle L_N, so that $H^0(X_N, L_N)$ carries all the highest weight vectors of $H^0(X, L)$. In other words, as T-modules:

$$H^0(X_N, L_N) \simeq H^0(X, L)^{N^+}.$$

The orbifold X_N naturally can be viewed as the compactification of the quotient of X/N^+. Therefore this compact model of X carries the same amount of information as X, in terms of understanding the induced representation.

To X_N obviously one can apply the fixed point formula, a certain generalization of Atiyah-Bott-Segal-Singer results to orbifold, and get informations about the T-equivariant Riemann-Roch $\mathrm{RR}(X_N, L_N)$, thus the structure of $H^0(X_N, L_N)$. But compactification involves adding certain locus, and additional fixed points sets on the locus. To understand those new fixed points is not an easy issue, even when dealing with finite dimensional groups, e.g. the compactification of symmetric spaces.

However here we will find a solution to resolve this problem here, utilizing the affine Weyl group W^{aff}. As it will be shown by examples, this solution is the best one can hope for.

1.4. **Description of the main result.** The main result in this paper does not require X is complex, although that situation motivates the construction of X_N and later consideration.

Let $Y = G \times_T X_N$, the line bundle L_N induces L_Y on Y. If the original line bundle is of level $k \in \mathbb{Z}$ (or the moment map μ is of level k), which means the central part of $\widetilde{LG}$, S^1, acts on L with character k, the following function on T will be uniquely determined by its restriction to $\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}$:

$$\sum_{w \in W} w \frac{1}{D} \mathrm{RR}(X_N, L_N) = \mathrm{RR}(Y, L_G).$$

The function $\mathrm{RR}(Y, L_G)$ can be defined directly from (X, L). It is given by

$$\sum_{a \in P_+^k} \mathrm{RR}(\mathcal{M}_a, L_a)\chi_a$$

where $\mathcal{M}_a$ is the reduced space of X at a, L_a is the line bundle induced from L, χ_a is the G-character function of the highest weight representation defined by a.

What geometric data are needed to determine this function? Before answering that question, let's motivate the discussion by first detailing the holomorphic case. As mentioned earlier, the quotient X/N^+, after throwing away some bad orbits, can be compactified by X_N. The compactification locus has its image given by the boundary of the affine chamber C. The chamber is a simplex if G is simple, and is a product of simplices if G is semi-simple. For generic X, X_N is an orbifold and strata in the compactification locus are in 1-1 correspondence with the sub-faces of C. Particularly interesting here are the strata $\{X_Q\}$ whose images are on the affine wall C^{aff}. Each X_Q has a corresponding subvariety in $Y = G \times_T X_N$, Y_Q. The collection $\{Y_Q\}$ is part of the compactifying strata in the G-space Y.

Each $\tau \in \{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}$ is a regular element in T. Thus its fixed point sets $\{V\}$ in X has images under μ in $\mathfrak{t}$. Each component V will induce a subvariety V_Δ in $Y_\Delta = wX_N \subset Y$. And it also induces a subvariety $V_Q = V_\Delta \cap Y_Q$. The collections $\{V_\Delta\}, \{V_Q\}$ will be used to determine $\mathrm{RR}(Y)(\tau)$.

We emphasize that in general τ has lots more fixed point sets than $\{V_\Delta\}, \{V_Q\}$ on the compactification strata in Y. Not all of the fixed points in the compactification are in the closure of the interior ones. So the important feature of the main result is that only the closure of the interior τ-fixed points $\{V_\Delta\}$, and their intersection with strata in the compactification $\{V_Q\}$, matter in determining $\mathrm{RR}(Y)(\tau)$. This feature manifests the underlying affine Weyl group symmetries, and is not known to hold in finite dimension.

Main Theorem 1. *At $\tau \in \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}$, the following holds:*

$$
\begin{aligned}
&\mathrm{RR}(Y)(\tau) \\
&= \sum_{a \in P_+^k} \mathrm{RR}(\mathcal{M}_a, L_a)\chi_a(\tau) \\
&= \sum_{V_\Delta} \Big(\int_{V_\Delta} \frac{\mathrm{Td}(V_\Delta)\mathrm{Ch}(L_{V_\Delta})}{\det_{\mathrm{nor}(V_\Delta, Y)}(1 - t^{-1}e^{-\Omega})} \\
&\qquad + \sum_{V_Q \subset V_\Delta} \frac{1}{|W_Q^{\mathrm{aff}}||I_{V_Q}|} \sum_{t \in \tau I_{V_Q}} \int_{V_Q} \frac{\mathrm{Td}(V_Q)\mathrm{Ch}\left(L_{V_Q} \oplus \Lambda^{\max}\mathrm{nor}(V_Q, V_\Delta)|_{V_Q}\right)}{\det_{\mathrm{nor}(V_Q, Y_Q)}(1 - t^{-1}e^{-\Omega})} \Big)(\tau)
\end{aligned}
$$

where $W_Q^{\mathrm{aff}} \subset W^{\mathrm{aff}}$ is the subgroup preserving Q, I_{V_Q} is the isotropy group associated with V_Q and τI_{V_Q} is the set of liftings of τ. Furthermore, each integral above can be localized to the T-fixed point sets F in V_Δ, V_Q respectively to yield:

$$
\mathrm{RR}(Y)(\tau) = \sum_{\{F|\mu(F) \in W(C^{\mathrm{int}})\}} \mathrm{FC}(F)(\tau) + \mathcal{R}(\tau) \tag{1.2}
$$

where

$$
\begin{aligned}
\mathrm{FC}_F(\tau) &= \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{\det_{\mathrm{nor}(F,Y)}(1 - t^{-1}e^{-\Omega})}(\tau); \\
\mathcal{R}(\tau) &= \sum_{\{F|\phi(F) \in W(C^{\mathrm{aff}})\}} \frac{1}{|W_\phi^{\mathrm{aff}}||I_F|} \sum_{t \in \tau I_F} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}\left(L_F \oplus \Lambda^{\max}\mathrm{nor}(Y_Q, Y)_F\right)}{\det_{\mathrm{nor}(F, Y_Q)}(1 - t^{-1}e^{-\Omega})}(\tau).
\end{aligned}
$$

where C^{int} is the interior of C, W_ϕ^{aff} is the subgroup of W^{aff} preserving $\phi(F)$, and I_F the isotropy group of F.

Remark: 1). Similar to an earlier comment, the interesting feature in the second expression above is that only those F on the intersection $V_\Delta \cap Y_Q$ matters. Other T-fixed points on the compactification locus Y_Q do exist and there are lots of them, but they do not contribute to $\mathrm{RR}(Y)(\tau)$ as it will be shown.

2). The presence of $I_F, \tau I_F, I_{V_Q}, \tau I_{V_Q}$ in fixed point formula is a common feature in orbifold setting, this has been known for a while.

Consequences on $\widetilde{LG}$-modules are given in Section 10.

1.5. **Riemann-Roch of the reduced spaces.** The previous result provides a way of computing the Riemann-Roch numbers of the reduced space $\mathcal{M}_a = \mu^{-1}(a)/(LG)_a$ via certain fixed point sets.

Corollary 1.1.

$$
\begin{aligned}
&\mathrm{RR}(\mathcal{M}_a, L_a) = \\
&\frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{\tau \in \{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}} \chi_{\bar{a}}(\tau) D^2(\tau) \Big(\sum_{\{F|\mu(F) \in kW(C^{\mathrm{int}})\}} \mathrm{FC}(F)(\tau) + \mathcal{R}(\tau) \Big)
\end{aligned}
$$

where $\bar{a} = w_L(-a)$ with w_L being the longest element in W (or the highest weight in the contragredient representation to the one defined by a).

Remark: The above extends Verlinde's formula for moduli space of flat connections over Riemann-surfaces.

If X is holomorphic and $H^i(\mathcal{M}_a, L_a) = 0$ for $i > 0$, then the above formula gives the multiplicities of the irreducible component with highest weight (a, k) for the $\widetilde{LG}$ representation on $H^0(X, L)$.

In finite dimension, the Riemann-Roch number of the reduced space can also be expressed in terms of the fixed points. The expression is obtained only recently as an application of Atiyah-Bott-Singer-Segal's formula and the recent in [M].

1.6. **What's in the proof?** The theorem is proved based on two main ingredients, both utilize the Weyl and affine Weyl group symmetries.

After applying directly fixed point formula for orbifolds to Y, one ends up with many terms of contributions from the fixed point sets on the compactifying locus. There are two kinds of them, those of the first kind have their images on $W(\partial C \setminus C^{\text{aff}})$, we prove using Weyl group symmetry that their contribution amounts to 0. The second kind are those with images on $W(C^{\text{aff}})$. Their contribution to the equivariant Riemann-Roch also amounts to 0, provided we restrict them as functions on T to the subset $\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P^k_+\}$, and each term has no pole on the subset. This is proved using affine Weyl group symmetries.

We shall refer to the above phenomenon as cancellation, it is based on the fundamental formula of Section 6 and several identies proved in Section 7. Also it requires detailed analysis of the fixed points on the compactification locus.

The second ingredient starts with the surgery formula in Section 12, it enables us to deal with the second kind of fixed points when poles are present. Their contribution is encoded in the function $\mathcal{R} : \{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P^k_+\} \to \mathbb{C}$. The exact expression of $\mathcal{R}$ is determined using transformation rule of the affine Weyl group acting on the Riemann-Roch integrand, together with symplectic cuts. The calculation is rather elaborate.

The construction of the space X_N, was first done in 1993 prior the symplectic cuts. It shares certain similarity with the symplectic cuts, except the cuts are made along the degenerated parts of the two form $\omega|\mu^{-1}(\mathfrak{t})$. The resulting space is only symplectic outside the inverse images of ∂C. The surgery formula for this kind of cuts is quite different from that using symplectic cuts, as shown by Prop. 12.1.

The cancellation mentioned earlier has a consequence called twin pair construction. It says that for a generic compact symplectic manifold with a Hamiltonian G-action, there is a different G-orbifold, with identical equivariant Riemann-Roch.

1.7. **What is ahead?** As mentioned earlier, applications will be given in a forthcoming paper.

In a separate paper, the result here will be improved so that all symplectic LG-manifolds, with compact quotient X/LG, will be covered. The present results assume the generic condition that

$$\mu(X) \subset l\mathfrak{g} \times \{k\} \subset \widetilde{l\mathfrak{g}}$$

is transversal to $\mathfrak{t}$.

1.8. **Organization of the paper.** In section 2, we discuss the construction of X_N. The paper [C1] emphasizes on the holomorphic aspects of X_N while here the construction builds around the symplectic structure. Several new results are presented here, including the existence of LG-invariant almost complex structures

on a class of LG-manifolds. The existence of T-invariant almost complex structure on X_N is proved as well, which is not trivial considering that the symplectic form on X_N is degenerate.

Section 3 contains the description of the fixed point sets, and their stratification which is a must since X_N is an orbifold. And the fixed point formulas on orbifolds rely not only on the fixed points but their stratification as well. Section 4 includes the computations of weights of the induced T-action on the normal bundles to the fixed point sets and their lower strata, while Section 5 computes the curvatures of various components of the normal bundles.

The root of the cancellation is the the fundamental formula presented in Section 6. I proved this formula in 1994. The original proof was based on comparison of two different compactifications of $G^{\mathbb{C}}$. Both have the same Riemann-Roch but have different fixed points. Hence one yields an identity, then by induction on the rank of the group, one proves this formula. The result in Section 12 generalizes this 'twin pair' construction. The present proof is simplified by applying an identity which can be found in [M].

Another important component in proving the cancellation is described in Section 7. Section 8 addresses a complication which occurs when $G \neq SU(n)$. For those groups, the affine alcove may not be a simple simplex, with respect to the weight lattice. In constructing X_N, this fact introduces addition orbifold singularities. That section describes the extra components of the isotropy groups associated with the orbifold singularities.

Section 9 provides a way of computing the push-forward of certain cohomology classes on a fibration whose fibers are homogeneous spaces like K/T. Several integration formulas there can be viewed as localization formulas for families.

In section 10, we discuss the relations between T-spaces and G-spaces, characters of T-modules and G-modules. Several consequences of the main theorem are proved.

Section 11 proves the main cancellation which has a couple of consequences discussed in Section 12, including the 'twin pair'. The proof of the main result is completed in the last section after we find an expression for the remainder term.

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2. Basic properties of the variety X_N

Many properties presented here were proved in [C1], with the exception of the existence of a LG-invariant almost complex structures on X, and a T-invariant one on X_N. In [C1], X is assumed to be holomorphic, therefore it is not necessary. We list the important properties for $l\mathfrak{g}, \mathbb{X}, X_N$ here.

2.1. **Basics of affine Lie algebra.** Let $\mathfrak{g}$ be the Lie algebra of G and T be a maximal torus with $\mathfrak{t}$ as its Lie algebra. The discussion works without much modification for semi-simple Lie groups. Let LG be the loop group associated with G, and $l\mathfrak{g} = \mathrm{Lie}LG$.

For functions on S^1, we use the $t \in [0,1]$ to parameterize them. The Fourier series components are $\{e^{2\pi int}\}$. On $l\mathfrak{g}$, there is the following well defined form in terms of the invariant non-degenerate form $(\cdot|\cdot)$ on $\mathfrak{g}^{\mathbb{C}}$:

$$(a|b) = \int_0^1 (a(t)|b(t))dt \in \mathbb{R}.$$

It induces a symplectic form on $l\mathfrak{g}/\mathfrak{g}$:

$$B(a,b) = (a'|b) = \int_0^1 (a'(t)|b(t))dt$$

where $a' = da/dt, t \in [0,1]$. The form is degenerate when restricted to the constant $\mathfrak{g}$. We choose the form on $\mathfrak{g}$ with the condition that $(\theta^{\mathrm{v}}|\theta^{\mathrm{v}}) = 2$ or equivalently $(\theta|\theta) = 2$, where θ^{v} is the coroot corresponding to the highest root θ of $\mathfrak{g}$. As pointed out in [PS, p. 46], the associated form B defines the smallest integral class on LG.

The affine Lie algebra based on $\mathfrak{g}$, $\mathfrak{g}^{\mathrm{aff}}$, is defined as

$$\mathfrak{g}^{\mathrm{aff}} = l\mathfrak{g} \oplus \mathbb{R}d \oplus \mathbb{R}K,$$

where K is the central element and $d = d/dt$ is the differentiation. The Lie bracket is

$$[\xi + \lambda d + cK, \eta + \lambda' d + c'K] = [\xi, \eta] + \lambda d\eta - \lambda' d\xi + B(\xi, \eta)K.$$

The central extension of $l\mathfrak{g}$, $\widetilde{l\mathfrak{g}}$, is given by $\mathfrak{g}^{\mathrm{aff}} = l\mathfrak{g} \oplus \mathbb{R}K$. It is the Lie algebra of $\widetilde{LG}$ which is a circle bundle over LG, whose existence of $\widetilde{LG}$ is proved in [PS].

The Lie algebra dual $\mathfrak{g}^{\mathrm{aff}*}$ is given by

$$\mathfrak{g}^{\mathrm{aff}*} = l\mathfrak{g}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0.$$

The bilinear form $(\cdot|\cdot)$ extends to $\mathfrak{g}^{\mathrm{aff}}$, $\mathfrak{g}^{\mathrm{aff}*}$, so that it is invariant. Its restriction to the 2-dim subspace $\mathbb{R}d \oplus \mathbb{R}K$, is of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. See [K, Ch. 6] for more details. Continue to use ν for the map $\mathfrak{g}^{\mathrm{aff}} \to \mathfrak{g}^{\mathrm{aff}*}$ defined by $(\cdot|\cdot)$. The bilinear form gives

$$\nu(l\mathfrak{g} \oplus \mathbb{R}d) = l\mathfrak{g}^* \oplus \mathbb{R}\Lambda_0.$$

The simple roots of $\mathfrak{g}^{\mathrm{aff}}$ consists of simple roots of $\mathfrak{g}$ together with $\alpha_0 := \delta - \theta$. Suppose that $\{E_i', F_i'\}$ are the Chevalley basis of $\mathfrak{g}$, let $E_0', F_0' \in \mathfrak{g}_\theta^{\mathbb{C}}$, so that

$$[E_0', F_0'] = -\theta^{\mathrm{v}},$$

where $-E_0$ is the Chevalley involution of F_0', define

$$e_0 = z \otimes E_0', \quad f_0 = z^{-1} \otimes F_0'.$$

For $su(2)$, the pair is

$$\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix}.$$

The pair e_0, f_0 together with $e_i = E_i', f_i = F_i'$, $1 \le i \le l$ generate $l\mathfrak{g}^{\mathbb{C}}$.

The positive affine roots $\gamma \in \Delta_+(\mathfrak{g}^{\text{aff}})$ are

$$\{n\delta \pm \alpha | n \ge 1, \alpha \in \Delta_+(\mathfrak{g})\} \cup \Delta_+(\mathfrak{g}) \cup \{n\delta | n \ge 1\}$$

and accordingly the basis of the root spaces are $E_\gamma = z^n E'_{\pm\alpha}, E'_\alpha$ or $z^n h_\alpha$, where $\{E'_\alpha\}$ is the root space basis of $\mathfrak{g}$. The basis of the compact form are

$$\{x_\gamma = e_\gamma - f_\gamma, y_\gamma = i(e_\gamma + f_\gamma)\}$$

where $\gamma > 0$. The standard complex structure $J^{l\mathfrak{g}}$ on $l\mathfrak{g}/\mathfrak{t}$ which inherits from the map $n^+ \to l\mathfrak{g}/\mathfrak{t} \simeq l\mathfrak{g}^{\mathbb{C}}/(n^- + \mathfrak{t}^{\mathbb{C}})$ has this description:

$$J^{l\mathfrak{g}}(x_\gamma) = y_\gamma, \quad J^{l\mathfrak{g}}(y_\gamma) = -x_\gamma. \tag{2.1}$$

By direct computation, for $h \in \mathfrak{t}$, we obtain

$$\begin{aligned} &x'_\gamma = 2\pi n y_\gamma, \quad y'_\gamma = -2\pi n x_\gamma; \\ &[h, x_\gamma] = (-1)^{\text{sign}(\alpha)}\alpha(h)/i y_\gamma = -(-1)^{\text{sign}(\alpha)} i\alpha(h) y_\gamma; \\ &[h, y_\gamma] = -(-1)^{\text{sign}(\alpha)}\alpha(h)/i x_\gamma = (-1)^{\text{sign}(\alpha)} i\alpha(h) x_\gamma. \end{aligned} \tag{2.2}$$

Hence for $\xi = \sum a x_\gamma + b y_\gamma$,

$$\begin{aligned} &(J\xi)' + [h, J\xi]) = -\sum (2\pi n - (-1)^{\text{sign}(\alpha)} i\alpha(h))(a x_\gamma + b y_\gamma); \\ &((J\xi)' + [h, J\xi])|\xi) = -\sum (2\pi n + (-1)^{\text{sign}(\alpha)}\alpha(h)/i)(|a|^2|x_\gamma|^2 + |b|^2|y_\gamma|^2). \end{aligned} \tag{2.3}$$

Now suppose that $\frac{1}{2\pi i}h$ is in the affine alcove C, then

$$n + (-1)^{\text{sign}(\alpha)}\frac{1}{2\pi i}\alpha(h) =< n\delta \pm \alpha, \frac{d+h}{2\pi i} >\ge 0,$$

because all the positive roots are positive on $C + \frac{d}{2\pi i}$, (our definition of d differs from that in [K], hence the extra constant). Therefore $((J\xi)' + [h, J\xi])|\xi) \le 0$. Let $\Omega(\xi, \eta) = (\xi' + [h, \xi]|\eta)$, earlier calculation shows that

Lemma 2.1. *The form $\Omega(\cdot, J\cdot)$ is semi-positive definite.*

2.2. **Adjoint and co-adjoint action.** Let $\xi, \eta \in l\mathfrak{g}$, then $[\xi + ad, \eta] = [\xi, \eta] + ad\eta$ which integrates to

$$g^{-1}(\xi + ad)g = \mathrm{Ad}_{g^{-1}}\,\xi + ag^{-1}dg + ad \in \mathfrak{g}^{\text{aff}}.$$

The linear map $\nu : \mathfrak{g}^{\text{aff}} \to \mathfrak{g}^{\text{aff}*}$ defined by $(\cdot|\cdot)$ has the effect that $\nu(\xi + ad) = \nu(\xi) + a\Lambda_0$, therefore the adjoint action by g is given by

Definition 2.1. *The adjoint action on $h + a\Lambda_0$ is*

$$ghg^{-1} + ag\frac{d}{dt}g^{-1} + ad.$$

The level of $h + a\Lambda_0$ is a, thus the adjoint action preserves the level. In case of level 1, the projection to the $l\mathfrak{g}$ part is exactly the gauge transformation.

2.3. **Topology on LG.** Since the group LG we are dealing with is the mapping space $Map(S^1, G)$, there is the question as to which norm is used for completion. The norms defined by $B(\cdot, J^{l\mathfrak{g}}\cdot), \Omega(\cdot, J^{l\mathfrak{g}}\cdot)$ are not strong enough. The weakest norm which is sensible geometrically, is the H^1-norm. We can use other Hilbert metrics or Banach metrics as long as they are stronger than H^1-norm.

Convention: The convention here is to let $l\mathfrak{g} = H(S^1, \mathfrak{g})$ be the space of maps completed under the norm of choice, and LG be the corresponding group. Set $l\mathfrak{g}^0$ to be the set with one degree less of derivative, i.e.,

$$l\mathfrak{g}^0 = H'(S^1, \mathfrak{g}) = \{h' | h \in l\mathfrak{g}\} \oplus \mathfrak{g}.$$

It is another completion of $C^\infty(S^1, \mathfrak{g})$, so that the map

$$h \in l\mathfrak{g} \mapsto h' \in l\mathfrak{g}^0$$

is Fredholm and bounded.

We will simply use $l\mathfrak{g}$ for $l\mathfrak{g}^0$, just keep on mind that the target of μ consists of elements with one degree less of differentiablity.

2.4. **Loop group actions and the assumptions H1, H2.** Let X be a Banach manifold with a differentiable action by LG,

$$\mu : X \to l\mathfrak{g} \oplus \mathbb{R} \simeq l\mathfrak{g}^* \oplus \mathbb{R}\Lambda_0$$

be a moment map associated with a symplectic 2-form ω on X. The isomorphism $\simeq$ is defined by the restriction of ν. The moment map is equivariant with respect to the LG-action on X and the adjoint action on $l\mathfrak{g}^* \oplus \mathbb{R}\Lambda_0$,

Remark: The Banach norm on TX does not have to be invariant; and the positive definite form $\omega(\cdot, J\cdot)$, in general, defines a topology weaker than the Banach norm, for any compatible almost complex structure on the tangent space.

Definition 2.2. *μ is of level $k \in \mathbb{Z}_+$ if $\mu(X) \subset l\mathfrak{g} \times \{k\}$.*

Remark: The topology on $l\mathfrak{g} \oplus \mathbb{R}$ as described makes the co-adjoint action a bounded map.

The following assumptions will be made:

H1: μ is proper with aforementioned topology.

H2: μ is transversal to $\mathfrak{t} \times \{k\}$ in $l\mathfrak{g} \times \{k\}$.

The first one is essential and is equivalent to the compactness of X/LG, and the second is technical.

Assuming **H1** and **H2**, then $X_\mathfrak{t} = \mu^{-1}(\mathfrak{t} \times \{k\})$ is a finite dimensional submanifold. It is not symplectic, $\omega|X_\mathfrak{t}$ has serious degeneracy. And it may not even be orientable. Whenever the stabilizer of $\mu(p)/lp$ in $l\mathfrak{g}$, $(l\mathfrak{g})_{\mu(p)}$, has a semi-simple part, $\omega|_{T_pX_\mathfrak{t}}$ is null on $(l\mathfrak{g})_{\mu(p)}/T$.

2.5. **Toric bundle $\mathbb{X}$ over LG/T.** It is a fundamental fact that the affine Weyl group W^{aff} acts on $\mathfrak{t} \times \{k\}$, the quotient domain of the action is given by a simplex $k(C, 1)$ where C is the affine alcove of $\mathfrak{g}$. One can also consider the action by W^{aff} on the dual space $\mathfrak{t}^* \times \{k\}$, the quotient domain is kC^* with C^* given by

$$C^* = \{\lambda | \, (\alpha|\lambda) \geq 0, (\theta|\lambda) \leq 1\} = \{\lambda | < \alpha^{\text{v}}, \lambda > \geq 0, < \theta^{\text{v}}, \lambda > \leq 1\}$$

where $\alpha^{\text{v}}, \theta^{\text{v}}$ are the coroots. The above descriptions of C, C^* are the duals of each other, through the map $\nu : \mathfrak{t} \to \mathfrak{t}^*$.

When there is no confusion, we will not distinguish between C, C^*.

The simplex C^* is not simple with respect to the weight lattice, the edges do not form a base of the weight lattice of $\mathfrak{t}$. In fact the edges are given by

$$\Lambda_i/a_i^{\vee}, \quad i = 1, ..., l$$

where l is the rank of $\mathfrak{g}$, $\{\Lambda_i\}$ is the set of fundamental weights of $\mathfrak{g}$ and $\{a_i^{\vee}\}$ are the labels in the dual Dynkin diagram. More on this can be found in [C1].

From the theory of toric varieties, we know that there is an orbifold toric variety $X_{\mathfrak{g}}$ and an orbifold line bundle L_N associated with C. The pair is the quotient of $\mathbb{C}P^l, H$ by a finite group, where H is the hyperplane line bundle. The details are in [C1].

Given $X_{\mathfrak{g}}$, we can associate with it a toric bundle over LG/T:

$$\mathbb{X} = LG \times_T X_{\mathfrak{g}}, \quad (gt, z) \simeq (g, tz).$$

The quotient is well defined since T is compact and the action is free of fixed points. The group LG has the same topology as described earlier.

The construction of $\mathbb{X}$ given here dated back to 1993.

There are many nice characteristics about $\mathbb{X}$, we list a few needed later.

On $\mathbb{X}$, there is an action by LG and T respectively, the two actions commute. The actions by LG, T satisfy the Hamiltonian conditions with moment maps given by

$$\mu_{\mathbb{X}} : \mathbb{X} \to l\mathfrak{g}^0 \times \{1\}, \quad \phi : \mathbb{X} \to \mathfrak{t}.$$

Let $\tilde{\phi} = (\phi, 1) \in \mathfrak{t} \times \{1\}$. Then

$$\mu_{\mathbb{X}}([g, z]) = \widetilde{\mathrm{Ad}}_g(\tilde{\phi}(z))$$

that the above is independent of the choice of (g, z) in $[g, z]$ is evident. Let $T_{[I,z]} = l\mathfrak{g}/\mathfrak{t} \oplus T_z X_{\mathfrak{g}}$. The 2-form on $\mathbb{X}$ can be described as

$$\omega_{\mathbb{X}}|_{[I,z]}((\xi, a), (\eta, b)) = (\xi' + [\phi(z), \xi]|\eta) + \omega_{X_{\mathfrak{g}}}(a, b) \quad a, b \in T_z X_{\mathfrak{g}} \quad \xi, \eta \in l\mathfrak{g}/\mathfrak{t}.$$

The form is degenerate whenever $\phi(z)$ hits the boundary of the affine alcove C. The null space is generated by $(LG)^{\mathrm{ss}}_{\tilde{\phi}}/T$ where $(LG)_{\tilde{\phi}}$ is the stabilizer of $(\phi, 1)$. The complex structure J is defined as: $J|_{l\mathfrak{g}/\mathfrak{t}} = J^{l\mathfrak{g}}$ while on $T_z X_{\mathfrak{g}}$, it is given by the original one on $T_z X_{\mathfrak{g}}$. The choice of this complex structure is due to the following reasons:

1). The action by $t \in T$ on the left is

$$t(g, z) = (tg, z) \simeq (tgt^{-1}, tz).$$

2). We want highest weight modules, rather than lowerst weight ones.

2.6. **The variety X_N.** Reverse the complex structure on X, so that $-\mu$ is the moment map, and $-\omega$ is compactible with the complex structure $-J$.

The variety X_N which is important to our study can now be described as follows:

$$X_N = (\Psi^{-1}(0)/LG = \{(p, q)|\mu(p) = k\mu_{\mathbb{X}}(q)\}/LG,$$

where $\Psi = -\mu + k\mu_{\mathbb{X}} : X \times \mathbb{X} \to l\mathfrak{g}^0$ is the moment map associated with the diagonal action by LG on the product space, with the 2-form $-\omega + \omega_{\mathbb{X}}$.

The sign here is chosen so that no inversion of the complex structure on $\mathbb{X}$ is necessary, this way we still get the highest vectors in the end.

Notice the level of Ψ is 0. Set

$$Y_C = \mu^{-1}(kC \times \{k\}),$$

by **H1,H2** it is a compact manifold with boundary.

The following shows that the toric variety can be used to close the gash which is the boundary of Y_C.

Proposition 2.1. *The space* X_N *is*

$$\{(p,q) \in Y_c \times X_{\mathfrak{g}} |\, \mu(p) = k\tilde{\phi}(q)\}/T.$$

The proof is simple, since each pair (p,q) with $\Psi(p,q) = 0$ can be conjugated to $(p',q') \in Y_c \times_T X_{\mathfrak{g}}$ with $\mu(p') = k\mu_{\mathbb{X}}(q') \in k(C,1)$; the pair is unique up to T.

From this description, considering the assumptions **H1, H2**, it is clear that X_N is a compact orbifold. The claim that X_N is holomorphic whenever X is complex is of more subtle nature, it is proved in [C1].

Because T, LG commute on $\mathbb{X}$, the action by T descends from the product $X \times \mathbb{X}$ to X_N. So does the moment map ϕ. The form $-\omega + \omega_{\mathbb{X}}$ when restricted to $\Psi^{-1}(0)$ is invariant under the LG-action, thus it descends down to a form on X_N. Denote it by ω_N. The pair ϕ, ω_N satisfy the conditions for Hamiltonian action, though ω_N is degenerate.

When X is not complex, we will see there is an T-invariant almost complex structure J on X_N, such that $\omega_N(\cdot, J\cdot)$ is semi-positive definite. If ω_N on X_N is symplectic, the existence of such a J is well known. For degenerate ω_N, the existence is not automatic.

2.7. **The existence of an LG-invariant J on X.** The existence result only needs assuming **H1**.

Step 1: Existence of a positive bilinear LG-invariant form on X.

Let $\{v\}$ be the vertices of $k(C,1)$. There are $l+1$ of them where l is the rank of $\mathfrak{g}$. For each v, let C^v be $k(C,1)$ after removing the face opposite to v. Let W_v be the Weyl subgroup in W^{aff} generated by reflections with respect to walls passing v. It is well known that W_v is finite and is the Weyl group of $(LG)_v$ which stabilizes v under the co-adjoint action. Now set

$$O_v = \cup_{w\in W_v} wC^v,$$

which is an open set. It is the star-shaped region with center v if we view the images of C under W^{aff} as a triangulation of $\mathfrak{t} \times \{k\}$. E.g. when $\mathfrak{g} = su(2)$, $C = [0,1]$ while $O_0 = (-1,1), O_1 = (0,2)$.

Clearly $\cup_v O_v$ is an open cover of $k(C,1)$, and there exists a partition of unity $\{\psi_v\}$ subordinate to the covering, and ψ_v is invariant under W_v. Define

$$S_v = \mu^{-1}\big(\widetilde{\text{Ad}}_{(LG)_v}(O_v)\big); \quad \mathcal{S}_a = \mu^{-1}(\widetilde{\text{Ad}}_{LG} O_a).$$

Using the conditions on μ, we have:

$$\mathcal{S}_a = LG(S_a) = LG \times_{(LG)_a} (LG)_v(S_v),$$

and $\{\mathcal{S}_a\}$ is a LG-invariant open cover of X. Let g_a be an $(LG)_a$-invariant Riemannian metric on S_a. It exists, since $(LG)_a$ is compact and S_a is of finite dimension.

Next we define a positive bilinear LG-invariant form g_a^A on $\mathcal{S}_a$. Clearly

$$T_p\mathcal{S}_a = l\mathfrak{g}/(l\mathfrak{g})_a \oplus T_pS_a, \tag{2.4}$$

on the second factor there is already a Riemannian metric g_a. On the first one, the choice for J is $J^{l\mathfrak{g}}$. With this choice, we have the positivity of $\Omega(\xi, J\xi)$ as shown earlier.

The form $(\cdot|\cdot)$ is the restriction of the bilinear invariant form on $\mathfrak{g}^{\text{aff}} = l\mathfrak{g} + \mathbb{R}\delta + \mathbb{R}d$, (see [K, Ch. 7]). And $[J\eta, x] = J[\eta, x]$ which can be verified using the property that if

$$[E_\alpha, E_\beta = c_{\alpha,\beta} E_{\alpha+\beta},$$

then

$$[E_{-\alpha}, E_{-\beta} = -c_{\alpha,\beta} E_{-\alpha-\beta}.$$

Thus g_a^A is invariant under the conjugation on $l\mathfrak{g}/(l\mathfrak{g})_a$ by $(LG)_\mu$.

Lemma 2.2. *The decomposition*

$$T_p\mathcal{S}_a = l\mathfrak{g}/(l\mathfrak{g})_a \oplus T_p S_a$$

is orthogonal with respect to $g_a^A, \omega_{\mathbb{X}}$.

Pf: It is orthogonal with respect to g_a^A by construction. To check that for $\omega_{\mathbb{X}}$, let $u \in T_p S_a$ and $\xi \in l\mathfrak{g}/(l\mathfrak{g})_a$, by the definition of moment map,

$$\omega_{\mathbb{X}}(u, \xi(p)) = (D_u\mu|\xi) = 0$$

where the last equality holds because $D_u\mu \in (l\mathfrak{g})_a \perp l\mathfrak{g}/(l\mathfrak{g})_a$ under $(\cdot|\cdot)$. QED

We can use the group action by LG to extend g_a^A to a LG-invariant form denoted by the same on $\mathcal{S}_a$. This is made possible due to

1). the invariance of g_a under $(LG)_a$ on S_a;

2). The complement of $T_p S_a$ in $T_p\mathcal{S}_a$ is $l\mathfrak{g}/(l\mathfrak{g})_a$ whose positive bilinear form as in Eq. (**??**) is invariant under the conjugation.

Now for each p in $\mathcal{S}_a$, the image under μ of the orbit $LG(p)$ meets O_a in a W_a-orbit on which ψ_a is constant, since ψ_a is W_a-invariant. Therefore, we can extend ψ_a to $\mathcal{S}_a$. The extension will still be denoted by ψ_a. Obviously, $\{\psi_a\}$ forms a partition of unity on X subordinate to $\{\mathcal{S}_a\}$.

Now clearly $g = \sum_i \psi_a g_a^A$ is an invariant positive bilinear form.

Remark: We do not call this form a Riemannian metric in order to avoid confusion, since the topology defined by g is weaker than that on X.

Step 2: Existence of an LG-invariant almost complex structure.

On finite dimensional space M, given an invariant symplectic form ω and a positive definite form g, there is an uniquely well defined almost complex structure. The exact construction of j in terms of g, ω is like this: the non-degenerate 2-form can be viewed as a linear map

$$\omega : TM \to T^*M,$$

the form g defines a map from $T^*M \to TM$ which is the inverse of the map $\nu : TM \to T^*M$ defined by the metric.

Denote $\nu^{-1} \cdot \omega$ by $f : TM \to TM$. Then it is straightforward that $f^* = -f$, the dual is taken with respect to g. Therefore, $-ff = (f^*f)$ is positive definite and has an uniquely defined positive definite square root, $(-f^2)^{1/2}$ which is a linear map. Clearly $f, (-f^2)^{1/2}$ commute, so the following is the desired j,

$$j = f(-f^2)^{-1/2}.$$

This almost complex structure is clearly invariant.

Two remarks: 1). The positive form $\omega(\cdot, j\cdot)$ may not coincide with g; 2). If g is given by $\omega(\cdot, J\cdot)$, then $j = J$.

Both of the above are easy to verify.

In the infinite dimensional situation, we refrain from defining J on TX by directly using the above because the definition of f^*f may cause problem.

In the particular situation we are facing, however, the map f^*f will be shown to be I except on a finite dimensional space. Let us examine the matter more closely.

Suppose $\mu = \mu(p) \in k(C,1)$ is covered by $\{O_b\}$ but no other O_a. By definition of O_a, it is clear that the stabilizer of μ in W^{aff}, W_μ, is

$$W_\mu = \cap_{\{b|O_b \ni \mu\}} W_b.$$

When μ is in the interior, $W_\mu = \{I\}$ and all of $\{O_a\}$ cover it; when $\mu = v$ is a vertex, only O_v covers it. Also μ is not in the support of ψ_a if $\mu \notin O_a$. Therefore at p, the positive form

$$g = \sum_{\{b|O_b \ni \mu\}} \psi_b g_b^A.$$

On $T_p\mathcal{S}_b = l\mathfrak{g}/(l\mathfrak{g})_b \oplus T_p S_b$, g_b^A agrees with the form

$$([\cdot, J\cdot]|\mu(p)) \tag{2.5}$$

on the first factor, for each b with $p \in O_b$. Therefore inside that tangent subspace, g agrees with the expression in Eq. 2.5 on

$$\cap_{p \in O_b} l\mathfrak{g}/(l\mathfrak{g})_b.$$

By the previous lemma, the finite dimensional subspace E_p generated by

$$\{T_p S_b \big| \mu(p) \in O_b\}$$

is orthogonal to $\cap_{\{b|O_b \ni \mu(p)\}} l\mathfrak{g}/(l\mathfrak{g})_b$ with respect to ω, g. Hence, ω and g are in diagonal form in the decomposition

$$T_p X = \cap_{\{b|O_b \ni \mu(p)\}} l\mathfrak{g}/(l\mathfrak{g})_b \oplus E_p.$$

Thus the map $f = \nu^{-1}\omega$ is of diagonal form. On the first factor, again by the previous lemma and the definition of g_b^A,

$$g = ([\cdot, J\cdot]|\mu),$$

therefore over that subspace

$$f = J = J^{l\mathfrak{g}}; \quad f(-f^2)^{-1/2} = f = J^{l\mathfrak{g}}.$$

On the finite dimensional subspace E_p, J is uniquely defined by the restriction of f. Hence it is invariant by $(LG)_\mu$. We can easily extend J to the whole X through invariance. Thus we have just proved the following

Proposition 2.2. *There is a LG-invariant almost complex structure on X such that $\omega(\cdot, J\cdot) = -(D_{J\cdot}\mu|\cdot)$ is positive definite.*

Remark: If X is complex, the above equality $J = -J^{l\mathfrak{g}}$ on $\cap_{\{b|O_b \ni \mu(p)\}} l\mathfrak{g}/(l\mathfrak{g})_b$ is not true in general.

2.8. **The almost complex structure J on X_N.** We will prove the existence of an almost complex structure on X_N by showing that there exists Hodge type decomposition for $T_{(p,q)}X \times \mathbb{X}$ where $\mu(p) = k\mu_{\mathbb{X}}(q)$. Such a decomposition is used in the finite dimensional situation to show the existence of almost complex structure (or holomorphic structure if the original manifold is) on the reduced space.

Suppose $\mu(p) = k\mu_{\mathbb{X}}(q) \in k(C,1)$. It will be shown in the next section, as a consequence of **H2**, that $\mathfrak{t}_p \cap \mathfrak{t}_q = 0$ which implies $(l\mathfrak{g})_p \cap (l\mathfrak{g})_q = 0$, since $(l\mathfrak{g})_q = \mathfrak{t}_q$.

The following is easier to show than in the case when X is holomorphic, for the almost complex structure J^X constructed earlier has very special property.

From now on, let X be equipped with the opposite of its usual complex structure. And continue to use the same one on $\mathbb{X}$ so that $-\omega(\cdot, J\cdot) + \omega_{\mathbb{X}}(\cdot, J\cdot)$ is non-negative definite.

Proposition 2.3. *The action by $l\mathfrak{g}$ on the product $T_{(p,q)}X \times \mathbb{X}, \Psi(p,q) = -\mu(p) + k\mu_{\mathbb{X}}(q) = 0$ has no kernel. There is a LG-invariant (acting diagonally) decomposition of the tangent space:*

$$T_{(p,q)}X \times \mathbb{X} = V_{(p,q)} \oplus l\mathfrak{g}_{(p,q)} \oplus Jl\mathfrak{g}_{(p,q)},$$

where $l\mathfrak{g}_{(p,q)}$ denotes the induced tangent vectors on the product, J is on the product and

$$V_{(p,q)} = \{(u,v) | D_{(u,v)}\Psi = D_{J(u,v)}\Psi = 0\}.$$

The decomposition is orthogonal with respect to

$$h(\cdot,\cdot) = -\omega(\cdot, J\cdot) + \omega_{\mathbb{X}}(\cdot, J\cdot)$$

on the product space.

Remark: The above decomposition would have been obvious had h defined a complete norm.

Pf: 1). Claim $D_{Jl\mathfrak{g}}\Psi$ is onto $l\mathfrak{g}^0$.

In the same notations as before, there is a decomposition of T_pX as $\cap_b l\mathfrak{g}/(l\mathfrak{g})_b \oplus E_p$ where E_p is of finite dimension. By construction, $J^X = J^{l\mathfrak{g}}$ and $J^{\mathbb{X}} = -J^{l\mathfrak{g}}$. Therefore, for $\xi \in \cap_b l\mathfrak{g}/(l\mathfrak{g})_b$,

$$D_{J\xi}\Psi = -D_{J^X\xi}\mu(p) + D_{J\xi}\mu_{\mathbb{X}}(q) = -2D_{J^{l\mathfrak{g}}\xi}\mu(p) = 2\widetilde{\mathrm{ad}}_{\mu(p)}J^{l\mathfrak{g}}\xi \in \cap_b l\mathfrak{g}/(l\mathfrak{g})_b,$$

where one degree of differentiability is lost due to the adjoint action. Because $J^{l\mathfrak{g}}$ preserves $l\mathfrak{g}$, and $\widetilde{\mathrm{ad}}_{\mu(p)}(\cdot)$ with $\mu(p) \in k(C,1)$ is an isomorphism on $\cap_{\{O_b|\mu(p)\in O_b\}} l\mathfrak{g}/(l\mathfrak{g})_b$, the above implies that $D\Psi$ is onto $\cap_b l\mathfrak{g}^0/(l\mathfrak{g})_b$. The orthogonal complement of $l\mathfrak{g}^0/(l\mathfrak{g})_b$ is the subspace generated by $(l\mathfrak{g})_b$ such that $\mu \in O_b$. If $D\Psi$ is not onto that finite dimensional complement, there is a η, with

$$(D\Psi|\eta) = 0.$$

In particular,

$$0 = D_{J\eta}\Psi(\eta) = -\omega(J\eta, \eta) + \omega_{\mathbb{X}}(J^{\mathbb{X}}\eta, \eta),$$

which forces $\eta(p) = 0$ and $\eta(q)$ in the null direction of $\omega_{\mathbb{X}}(\cdot, J^{\mathbb{X}}\cdot)$. The analysis on the null space of $\omega_{\mathbb{X}}(\cdot, J^{\mathbb{X}}\cdot)$ shows that $\eta \in (l\mathfrak{g})^{\mathrm{ss}}_{\mu}$, where $\mu = \mu(p) = k\mu_{\mathbb{X}}(q)$. But **H2** is equivalent to

$$(l\mathfrak{g})_p \cap [(l\mathfrak{g})_\mu, (l\mathfrak{g})_\mu] = 0,$$

a contradiction.

2). Given a tangent vector in the product, (u,v), from the claim there exists $\xi \in l\mathfrak{g}$ such that $D_{J\xi}\Psi = D_{J(u,v)}\Psi$. And choose $\eta \in l\mathfrak{g}$ to satisfy $D_{J\eta}\Psi = D_{(u,v)}\Psi$, then

$$(u',v') = (u,v) - J\eta(p,q) - \xi(p,q)$$

satisfies

$$D_{(u',v')}\Psi = 0; \quad D_{J(u',v')}\Psi = 0$$

because $D_\tau\Psi = -\widetilde{\mathrm{ad}}_\tau\mu(p) + \widetilde{\mathrm{ad}}^*_\tau\mu_{\mathbb{X}}(q) = 0, \forall\tau \in l\mathfrak{g}$. Therefore

$$(u',v') \in \ker D\Psi \cap J\ker D\Psi = V_p.$$

3). The isomorphism between V_p and the tangent space to the orbit $\ker D\Psi/l\mathfrak{g}$ holds due to the claim in Step 1. Since V_p is invariant under J, there exists J on TX_N. QED

3. The Fixed Point Set on X_N

3.1. **When a point is fixed by T.** Let $u \in X_N$, then $u = [p, q]$ with $(p, q) \in X \times \mathbb{X}$, the bracket denotes the equivalence class under the diagonal action by LG.

The point q being in $\mathbb{X}$ is itself an equivalent class, $q = [h, z]$ with $h \in LG$, $z \in X_{\mathfrak{g}}$. The equivalent class represented by q is the orbit by the diagonal T action on $LG \times X_{\mathfrak{g}}$. Here the T action is from the right on LG.

Both equivalence relations can be kept tracked of by introducing the following:

Definition 3.1. *For points in $X \times LG \times X_{\mathfrak{g}}$, define the equivalence relation:*

$$(p, h, z) \simeq (gp, ghs, sz) \tag{3.1}$$

for $g \in LG, s \in T$.

A triple defines a point in X_N iff $\mu_X(p) = k\widetilde{\mathrm{Ad}}_g(\tilde{\phi}(z))$ where $\tilde{\phi} = (\phi, 1)$ and $\widetilde{\mathrm{Ad}}$ is the adjoint action.

On $X \times LG \times X_{\mathfrak{g}}$, the diagonal action by LG on the first two factors and the diagonal T-action on the last two factors, commute with T acting on the last one. And the T-action acting on $X_{\mathfrak{g}}$ alone preserves the moment maps $\mu - \Phi$, therefore it descends to one on X_N.

We make a semi-canonical choice for $[p, q]$, so that $q = [I, z]$. Because $0 = \Psi(u) = \mu_X(p) - k\Phi(q)$, we have $\mu(p) \in k(\mathfrak{t}, 1)$. The ambiguity is due to the following:

$$(p, I, z) \simeq (s^{-1}p, I, sz), \quad \forall s \in T$$

as in Definition 3.1. Thus $(s^{-1}p, I, sz), s \in T$ represents the same point as (p, I, z) in X_N.

Lemma 3.1. *Let $\mathfrak{t}_z$ be the Lie algebra of the stabilizer $T_z \subset T$ of $z \in X_{\mathfrak{g}}$, $\mathfrak{t}_p$ be that for the point p in X. Then a point $u = [p, I, z]$ is fixed by T, iff that $\mathfrak{t} = \mathfrak{t}_z + \mathfrak{t}_p$.*

Pf: The condition is clearly sufficient. Suppose u is fixed by g, i.e. $(p, I, gz) \simeq (p, I, z)$, that is for some $h \in T$,

$$(p, I, gz) = (h^{-1}p, I, hz).$$

Thus

$$hp = p, \quad h^{-1}gz = z$$

for $h \in T$. So $h \in T_p$ and $h^{-1}g \in T_z$, and $g = h \cdot h^{-1}g \in$. Therefore, any $g \in T$ can be decomposed into a product of elements in T_p, T_z, hence $\mathfrak{t} = \mathfrak{t}_z + \mathfrak{t}_p$. QED

Assume (p, I, z) defines a point in X_N, then we have $\mu(p) = k\tilde{\phi}(z) = k(\phi(z), 1)$ by the definition of X_N.

Lemma 3.2. *The following holds:*

1.) $\mathfrak{t}_z = (l\mathfrak{g})^{\mathrm{ss}}_{\tilde{\phi}} \cap \mathfrak{t}$.

2). Under the assumption that the image of μ_X is transversal to $(\mathfrak{t}, k)$ in $(l\mathfrak{g}, k)$, $\mathfrak{t}_z \cap \mathfrak{t}_p = 0$.

Pf: Let C_μ be the smallest wall of ∂C containing $\phi(z)$, and let V_μ be the linear subspace in $\mathfrak{t}$ parallel to C_μ. Then $V_\mu^\perp = \mathfrak{t}_z$, which is a basic fact in toric variety theory, or symplectic geometry.

The affine Lie algebra based on $\mathfrak{g}$, $\mathfrak{g}^{\mathrm{aff}}$, is $l\mathfrak{g} + \mathbb{R}d + \mathbb{R}K$ as in [K]. The dual is $l\mathfrak{g} + \mathbb{R}\delta + \mathbb{R}\Lambda_0$. The simple roots are $\{\alpha_0, \alpha_1, ..., \alpha_l\}$ where $\{\alpha_1, ..., \alpha_l\}$ is the set of simple roots of $\mathfrak{g}$, and $\alpha_0 = \delta - \theta$ which acts on $(h, l) \in \mathfrak{t} + \mathbb{R}\Lambda_0 \subset l\mathfrak{g} + \mathbb{R}\delta + \mathbb{R}\Lambda_0$ as $l - \alpha(h)$. The boundary of the alcove C is defined by $\cap\alpha_i^{-1}(0)$.

The stabilizer of $\tilde{\phi}$ is the same as that of C_μ. And the stabilizer $(LG)_\mu$ of C_μ in LG is generated by $\mathfrak{t}$ and $l\mathfrak{g}_{\alpha_i}$ such that $\alpha_i(C_\mu) = 0$. Clearly $l\mathfrak{g}_\mu^{\mathrm{ss}} = \sum_{\alpha(C_\mu)=0} l\mathfrak{g}_\alpha$. We claim that

$$(\xi|\tilde{\phi}) = 0, \quad \forall \xi \in (l\mathfrak{g})_\mu^{\mathrm{ss}}.$$

This is clearly true for x_α, y_α in $l\mathfrak{g}_\alpha$, using the standard notation for $l\mathfrak{g}_\alpha^{\mathbb{C}} \simeq sl_2(\mathbb{C})$. As for the coroot $\alpha^{\mathrm{v}} = [x_\alpha, y_\alpha]$, it holds as well because $\alpha(\mu) = 0$ and

$$(\alpha^{\mathrm{v}}|\mu(p)) = (x_\alpha|[y_\alpha, \mu]) = \alpha(\tilde{\phi})(x_\alpha|x_\alpha) = 0.$$

Thus we have $l\mathfrak{g}_\mu^{\mathrm{ss}} \cap \mathfrak{t} = \sum_{\alpha_i(C_\mu)=0} \mathbb{R}\alpha_i^{\mathrm{v}}$, the right hand side are in $\mathfrak{t}_z$ from the orthogonal condition just proved. We also know that the $V_\mu^\perp$ is exactly spanned by those α^{v}, since C_μ as a subface is defined this way. Hence $\mathfrak{t}_z = l\mathfrak{g}_\mu^{\mathrm{ss}} \cap \mathfrak{t}$

2). The transversality condition is equivalent to

$$l\mathfrak{g}_p \cap [l\mathfrak{g}_\mu, l\mathfrak{g}_\mu] = \{0\},$$

but $l\mathfrak{g}_\mu^{\mathrm{ss}} = [l\mathfrak{g}_\mu, l\mathfrak{g}_\mu]$, and $\mathfrak{t}_p \subset l\mathfrak{g}_p$, therefore we have $\mathfrak{t}_p \cap [l\mathfrak{g}_\mu, l\mathfrak{g}_\mu] = \{0\}$ which implies that $\mathfrak{t}_p \cap \mathfrak{t}_z = 0$ since $\mathfrak{t}_z \subset [l\mathfrak{g}_\mu, l\mathfrak{g}_\mu]$. QED

Let T_p^0 be the connected component of I in T_p. By the construction of the toric variety $X_\mathfrak{g}$, it is easy to see that T_z is connected. If $\mathfrak{t}_p \oplus \mathfrak{t}_z = \mathfrak{t}$, naturally $(t_p, t_z) \in T_p^0 \times T_z \mapsto t_p t_z \in T$ is a covering map.

Proposition 3.1. *Let $\tilde{F}_p$ be the connected component of T_p^0-fixed point set in X containing p, and M_z be the connected component of T_z-fixed point set in $X_\mathfrak{g}$ containing z.*

Then the connected component containing (p, I, z) of T-fixed point set in X_N is given by $(\tilde{F}_p \times \{I\} \times M_z) \cap \Psi^{-1}(0)$.

Pf: The inclusion of the set in the connected component of the fixed point set by T is clear. To see the other direction, suppose (q, I, w) is in the connected component of (p, I, z) which is fixed by T, and suppose q is close to p, w is close to z. The stabilizers in T of q, w must be subgroups of T_p^0, T_z, which is well known. On the other hand, we have shown $\mathfrak{t}_q + \mathfrak{t}_w = \mathfrak{t}, \mathfrak{t}_q \cap \mathfrak{t}_w = 0$. Therefore the inclusion of $\mathfrak{t}_q \subset \mathfrak{t}_p$, $\mathfrak{t}_w \subset \mathfrak{t}_z$ are equalities instead. Thus, $\mathfrak{t}_p, \mathfrak{t}_z$ vanish at q, w respectively.

If (q, I, w) is connected in the fixed point set to (p, I, z) through a 1-parameter curve (q_s, I, w_s), with $q_0 = p, w_0 = z$ and $q_1 = q$, $w_1 = z$, then for small s, by the above argument, q_s, w_s are fixed by $\mathfrak{t}_p, \mathfrak{t}_w$. So they are in the connected components of p, z fixed by T_p, T_z respectively. The above argument shows also the set $\{s\}$ such that (q_s, I, w_s) is in the desired product is open. Obviously the set is also closed. Therefore, (q, I, w) is in the product. QED

3.2. **More about the T-fixed point set on X_N.** From earlier discussion, we have learned that $[p, I, z]$ is a fixed point of T on X_N iff $\mathfrak{t}_p \oplus \mathfrak{t}_z = \mathfrak{t}$. Let T_p^0 denote the connected component of T_p. To understand the structure of the fixed point set, we need the following:

Definition 3.2. *Suppose $\mathfrak{t}_z \neq 0$, i.e. $\phi(z) \in \partial C$, let*

$$\mathcal{K} = \{g \in (LG)_\mu \Big| \operatorname{Ad}_g t = t, \forall t \in T_p^0\}, \quad \mathcal{N} = \{g \in (LG)_\mu \Big| \operatorname{Ad}_g(T_p^0) = T_p^0\},$$

and $\mathrm{Lie}\mathcal{K}$, $\mathrm{Lie}\mathcal{N}$ *be their Lie algebras.*

Lemma 3.3. *The groups $(LG)_\mu$ and $\mathcal{K}$ are compact and connected.*

Suppose $(l\mathfrak{g})^{\mathrm{ss}}_\mu \cap \mathfrak{t}_p = 0$ which is true under the assumption that the image of μ is transversal to $\mathfrak{t}$, then $\mathrm{Lie}\mathcal{N} = \mathrm{Lie}\mathcal{K}$.

The weights of the $\mathfrak{t}_p$ action on $(l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}$ are non-trivial, therefore $\mathcal{K}$ is the largest connected group acting on $\tilde{F}_p$.

Pf: The assertion on $(LG)_\mu$ is well known. For $\mathcal{K}$, the argument is also standard but we include here anyway. Suppose $\mathcal{K}^0$ is the connected a component passing I. Let $g \in \mathcal{K} \setminus \mathcal{K}^0$, then $\mathrm{Ad}_g \mathcal{K}^0 = \mathcal{K}^0$. The group T is a maximal torus in $(LG)_\mu$ and is contained in $\mathcal{K}^0$. Therefore $\mathrm{Ad}_g T$ is a maximal torus. By the uniqueness of maximal torus under the adjoint action, there is a $h \in \mathcal{K}^0$ such that $\mathrm{Ad}_{hg} T = T$. Obviously $hg \in W(\mathcal{K}^0)$, so hg is contained in the semi-simple part of $\mathcal{K}^0$. Thus g is connected to I and $\mathcal{K} = \mathcal{K}^0$.

The adjoint action by T^0_p on $\mathrm{Lie}\mathcal{N} \setminus \mathrm{Lie}\mathcal{K}$ has those roots of $\mathrm{Lie}\mathcal{N}$ which are not roots of $\mathrm{Lie}\mathcal{K}$ as eigenvalues. Let α be one of them, then because $\mathfrak{t}_p$ is normalized by $\mathcal{N}$, the reflection element in the Weyl group r_α satisfies $r_\alpha(\mathfrak{t}_p) = \mathfrak{t}_p$. Using the definition of the reflection, and the fact that $\alpha(\mathfrak{t}_p) \neq 0$, one concludes the coroot $\alpha^{\mathrm{v}} \in \mathfrak{t}_p$. On the other hand $\alpha^{\mathrm{v}} \in \mathfrak{t}_z$, since $\mathfrak{t}_z$ is the Cartan subalgebra of the $\mathfrak{g}^{\mathrm{ss}}_\mu$ which contains $\mathrm{Lie}\mathcal{N}^{\mathrm{ss}}$. But $\mathfrak{t}_z \cap \mathfrak{t}_p = 0$, hence it is impossible that $\alpha^{\mathrm{v}} \in \mathfrak{t}_p$. So all the roots of $\mathrm{Lie}\mathcal{N}$ are those of $\mathrm{Lie}\mathcal{K}$. From this and the fact that $\mathfrak{t}$ is the Cartan sub-algebra of both groups, one concludes $\mathrm{Lie}\mathcal{N} = \mathrm{Lie}\mathcal{K}$.

Now the assertion that the action by $\mathfrak{t}_p$ on $l\mathfrak{g}_\mu/\mathrm{Lie}\mathcal{K}$ have non-trivial weights follows immediately. QED

3.3. Z and Z/T_z as fiber bundles.

Proposition 3.2. *Let X be the connected component containing p in $\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})$. Then the connected component of T-fixed point set on X_N passing $[p, I, z]$ is given by $Z/T_z \times \{z\}$.*

If ϕ is in the interior of C, $T_z = I$.

In case ϕ is on the boundary of C, Z/T_z admits a projection to an orbifold E with fiber a finite quotient of $\mathcal{K}/T$.

Pf: Suppose $[q, I, w]$ is in a same connected component as $[p, I, z]$ in X_N fixed by T. It follows from Prop. 3.1 that $q \in \tilde{F}_p$ and $w \in M_z$. A basic property of moment map restricted to fixed point set dictates that

$$\mu(\tilde{F}_p \cap \mu^{-1}(\tilde{\mathfrak{t}})) \subset \mu(p) + \mathfrak{t}_p^\perp, \quad \phi(M_z) \subset \phi(z) + \mathfrak{t}_z^\perp,$$

and $\mathfrak{t}_p^\perp \cap \mathfrak{t}_z^\perp = 0$ since $\mathfrak{t}_p \oplus \mathfrak{t}_z = \mathfrak{t}$. The condition $\mu(q) = k\tilde{\phi}(w) \in \tilde{\mathfrak{t}}$ then forces

$$\mu(q) = \mu(p) = k\tilde{\phi}(z) = k\tilde{\phi}(w).$$

As points in the toric variety $X_{\mathfrak{g}}$, the equality $\mu_{\mathbb{X}}(z) = \mu_{\mathbb{X}}(w)$ implies that $w = tz$ for some $t \in T$. Therefore $[q, I, w]$ has a representative of the form $(t^{-1}q, I, z)$ which defines a point in the claimed set. Thus the association of $[q, I, w]$ with a point in the quotient space $(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/T_z) \times \{z\}$ is 1-1 and onto.

As for the last assertion, we already know that $\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})$ admits the action of $\mathcal{K}$, since it commutes with T_p and preserves $\mu^{-1}(\tilde{\phi})$. Let $E = Z/\mathcal{K}$. The fiber of the projection

$$\pi : \left(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})\right)/T \to \left(\tilde{F}_p \cap \mu^{-1}(\tilde{\phi})\right)/KK = E$$

can be explicitly described. If $[q] \in E$ with $q \in \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})$, then the fiber is $\mathcal{K}(q) = \mathcal{K}/\mathcal{K}_q$. We know that $\mathcal{K}_q$ has T_p^0 as the maximal connected subgroup, since $\mathrm{Lie}\mathcal{K}_q$ has no semi-simple part from the condition $l\mathfrak{g}_q \cap [l\mathfrak{g}_\mu, l\mathfrak{g}_\mu] = 0$. Hence T_p^0 is a normal subgroup and $\mathcal{K}_q/T_p^0$ is a finite subgroup. Therefore the fiber is a finite quotient of the homogeneous space $\mathcal{K}/T$ by K_q/T_p^0. QED

The following obviously holds:

$$Z = \tilde{F}_p \cap \mu^{-1}(\tilde{\phi}) \longrightarrow \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/T \longrightarrow \tilde{F}_p \cap \mu^{-1}(\tilde{\phi})/\mathcal{K} = E. \tag{3.2}$$

Since T_p^0 fixes points on Z, the action by $\mathcal{K}$ is not effective on Z. In fact, we have

Lemma 3.4. *Define* $\mathfrak{t}^0$ *to be a complement of* $\mathfrak{t}_z \cap \mathrm{Lie}\mathcal{K}^{\mathrm{ss}}$ *in* $\mathfrak{t}_z$*, then*

$$\mathrm{Lie}\mathcal{K} = \mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \oplus \mathfrak{t}_0 \oplus \mathfrak{t}_p,$$

where $\mathrm{Lie}\mathcal{K}^{\mathrm{ss}}$ *is the semi-simple part of* $\mathrm{Lie}\mathcal{K}$*.*

Pf: It is known that $\mathfrak{t} \subset \mathrm{Lie}\mathcal{K}$, so $\mathrm{Lie}\mathcal{K} = \mathrm{Lie}\mathcal{K}^{\mathrm{ss}} + \mathfrak{h}$ with $\mathfrak{h} \subset \mathfrak{t}$. On the other hand, $[\mathrm{Lie}\mathcal{K}^{\mathrm{ss}}, \mathfrak{t}_p] = 0$ by definition of $\mathcal{K}$, hence $\mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \perp \mathfrak{t}_p$. But $\mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \subset (l\mathfrak{g})_\mu^{\mathrm{ss}}$ which has $\mathfrak{t}_z$ as its Cartan subalgebra. Therefore $\mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \cap \mathfrak{t} \subset \mathfrak{t}_z$. We already knew that $\mathfrak{t} = \mathfrak{t}_p \oplus \mathfrak{t}_z$, hence

$$\mathrm{Lie}\mathcal{K} = \mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \oplus \mathfrak{t}_p \oplus \mathfrak{t}_0$$

where $\mathfrak{t}_0 \subset \mathfrak{t}_z$ and is perpendicular to $\mathfrak{t} \cap \mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \subset \mathfrak{t}_z$. QED

Let $\mathcal{K}'$ be the group with $\mathrm{Lie}\mathcal{K} = \mathrm{Lie}\mathcal{K}^{\mathrm{ss}} \oplus \mathfrak{t}_0$ as its Lie algebra. This group acts effectively on Z, and T_z is its Cartan subgroup.

So the fiber over E is a finite quotient of $\mathcal{K}'/T_z$.

3.4. **Connections of the orbifold fiberations.** Let $\mathrm{Lie}\mathcal{K}' = \mathfrak{t}_z + \mathfrak{n}$ be the Cartan decomposition.

Fix a $\mathcal{K}'$-stable splitting of $TZ = T^H Z \oplus T^\perp Z$, where $T^\perp Z$ is tangent to the orbit by $\mathcal{K}'$ on Z. Such a splitting exists because the action by $\mathcal{K}'$ is locally free, therefore there is automatically the bundle $T^\perp Z$. The $\mathcal{K}'$-stable horizontal space can be obtained by an invariant metric on TZ.

Let $A : TZ \to T^\perp Z$ be the projection. There is a further decomposition:

$$T^\perp Z = T'Z + T''Z,$$

where $T'Z$ and $T''Z$ are the vertical subbundle generated by vectors in $\mathfrak{t}_z$ and $\mathfrak{n}$ respectively.

Lemma 3.5. *1). If* $(p, n) \in Z \times \mathfrak{n} \mapsto n(p) \in T_p'' Z$*, then the action by* T_z *on* $T''Z$ *induced from the action on* Z*, maps* (p, n) *to* $(pt, \mathrm{Ad}_t\, n), \forall t \in T_z$*.*

2). $T''(Z/T_z) = Z \times_{T_z} \mathfrak{n}$ *where the quotient is taken with the previous* T_z *action on* $T''Z \times \mathfrak{n}$*. And* $T(Z/T_z) = \pi^* TE \oplus T''(Z/T_z)$*, where* $\pi : Z/T \to E = Z/\mathcal{K}'$*.*

Pf: Let g_* denote the action on the tangent bundle. It is straightforward to verify that

$$\xi(gp) = g_*(\mathrm{Ad}_{g^{-1}}\, \xi)(p)), \forall g \in \mathcal{K}', \xi \in \mathrm{Lie}\mathcal{K}'. \tag{3.3}$$

Hence $(gp, \xi) \mapsto (p, \mathrm{Ad}_{g^{-1}}\, \xi)$. Or $(p, \xi) \mapsto (gp, \mathrm{Ad}_g\, \xi)$, which implies the first assertion after applying it to $\xi = n \in \mathfrak{n}$ and $g = t \in T_z$.

2). This is simply the decomposition of $T(Z/T_z)$ into horizontal and vertical parts. QED

Denote the map which identifies the vertical vectors with Lie$\mathcal{K}'$ by P. Combine with A, we obtain

$$PA : TZ \to \mathrm{Lie}\mathcal{K}'.$$

Furthermore $P = P' + P''$ with P', P'' take values in $\mathfrak{t}_z, \mathfrak{n}$ respectively. Obviously PA is a connection on Z.

Let u, v be vector fields in $T^H Z$ invariant under Lie$\mathcal{K}$, and let ξ, η be vectors induced by two elements from $\mathfrak{n}$.

The decomposition of the curvature given below will be useful later.

Lemma 3.6. *Suppose T_z acts on $\mathbb{C}$ with character λ, then there is a connection A_λ on the bundle $Z \times_{T_z} \mathbb{C} \to Z/T_z$ such that its curvature is given by $\lambda(B) + \lambda(R)$; the 2-forms B and R are $\mathfrak{t}_z$-valued, and satisfy the following*

$$B(u, v) = P' \cdot A([u, v]), \quad R(\xi, \eta) = P' \cdot A([\xi, \eta]).$$

Pf: On the trivial bundle $Z \times \mathbb{C}$, the action by Lie algebra of $\mathfrak{t}_z$ is given by $\mathcal{L}_t = (t(q), \lambda(t))$, for $t \in \mathfrak{t}_z$. The connection defined by $\nabla = d + \lambda(P'A)$ satisfies

$$\nabla_{t(q)} = d_{t(q)} + \lambda(P'A)(t(p)) = d_{t(q)} + \lambda(t) = \mathcal{L}_t$$

by the definitions of P, A. A connection, satisfying the above condition, descends to the quotient bundle $Z \times_{T_z} \mathbb{C}$. The curvature is given by $d(\lambda(P'A))$. For vector fields u, ξ, by invariance of u, we know that

$$[u, \xi] = 0, \quad A(u) = 0, \quad PA(\xi(q)) \in \mathfrak{n}$$

hence $P'A(\xi(q)) = 0$ and the differential of the 1-form, $d(\lambda(P'A))(u, \xi) = 0$. Thus the curvature has no term mixing $T^H Z$ and $T''Z$, and consists only of the vertical and the horizontal part. The one coming from the horizontal vector fields u, v is exactly

$$\lambda \cdot B(u, v) = \lambda \cdot P'A([u, v]) = \lambda \cdot PA([u, v]),$$

since $A(u) = A(v) = 0$ and $\lambda \cdot P'' = 0$. The one coming from the fiber, or the vertical directions, is $\lambda(PA([\xi(q), \eta(q)]) = \lambda(P'A([\xi(q), \eta(q)])$. This follows from $\lambda(\mathfrak{n}) = 0$ and $P'A(\xi(q)) = P'A(\eta(q)) = 0$, hence in calculating $dP'A(\xi, \eta)$ only the said term remains. QED

Fix a base point q, one may ask how the forms B, R transform along the fiber of π.

Lemma 3.7.

$$\lambda \cdot B(u, v)|_{gp} = \lambda(\mathrm{Ad}_g \, PA[u(q), v(q)]),$$
$$\lambda \cdot R(\xi(gq), \eta(gq)) = \lambda(\mathrm{Ad}_g[\xi(q), \eta(q)]).$$

Pf: Due to the invariance of u, v and the fact that g_* commutes with $[,]$, we have $[u(gq), v(gq)] = g_*([u(q), v(q)])$. The map g_* also commutes with A by the invariance of A, therefore

$$A[u(gq), v(gq)] = g_*(A([u(q), v(q)])).$$

Let $a = PA([u(q), v(q)]) \in \mathrm{Lie}\mathcal{K}'$, then its induced vector $a(q) = A([u(q), v(q)])$ by the definition of P. We know $g_*(a(q)) = \mathrm{Ad}_g\, a(gq)$, thus

$$
\begin{aligned}
PA[u(gq), v(gq)] &= Pg_*(A([u(q), v(q)])) \\
&= Pg_*(a(q)) \\
&= P\,\mathrm{Ad}_g(a)(gq) \\
&= \mathrm{Ad}_g(a),
\end{aligned} \tag{3.4}
$$

hence $\lambda(PA[u(gq), v(gq)]) = \lambda(\mathrm{Ad}_g\, PA[u(q), v(q)])$. For the same reason, we obtain

$$\lambda \cdot R(\xi(gq), \eta(gq)) = \lambda \cdot \mathrm{Ad}_g\, PA([\xi(q), \eta(q)]). \quad \text{QED}$$

This expression is crucial to a calculation in Section 9, because λR behaves as the moment map on the fiber which is a coadjoint orbit.

3.5. **Stratification of the fixed point set.** Unlike the smooth case, the fixed point formula for orbifolds requires the contributions of lower strata of the fixed point set. Where do the strata come from? They are present due to the local isotropy groups on the fixed point set Z/T_z.

To describe them locally, let $p \in Z$, T_p may not be connected. Denote the connected component of I by T_p^0. Let $I_p = T_p \cap T_z$, $I_p^0 = T_p^0 \cap T_Z$. They are finite groups since $\mathfrak{t}_p \cap \mathfrak{t}_z = 0$. When T_z acts on Z, it has I_p as its stabilizer. All $q \in Z$ are fixed by T_p^0, so the subgroup I_p^0 acts trivially on Z. The effective isotropy group on Z is I_p/I_p^0, though I_p^0 may have non-trivial action on the normal bundle and can not be ignored. Obviously the discussion is unnecessary if for all $p \in Z$, $T_p = T_p^0$.

For each $h \in I_p/I_p^0$, denote by Z_h its fixed points in Z. The collection $\{Z_h\}$ for $h \in I_p/I_p^0, \forall p \in Z$ form stratification of Z. And their quotient by T_z in $F = Z/T_z$ contribute to the fixed point formula computations, which is different from the smooth case.

Let $h \in I_p \setminus I_p^0$, let $Z_h = \{q \in Z | h(q) = q\}$. It is a submanifold. Clearly T_z acts on it, and the points there are fixed by h, T_p^0. Let $\mathcal{K}_h$ be the connected subgroup of $\mathcal{K}$ which commutes with h, i.e., $\mathrm{Ad}_k\, h = h, k \in \mathcal{K}_h$. Then $\mathcal{K}_h$ acts on Z_h, and it contains T.

In the language above, Z itself can be thought of as $Z_h, h \in I_p^0$.

3.6. **Lower stratum Z_h and Z_h/T_z as fiber bundles.** Obviously one has

Lemma 3.8. *The Lie algebra of $\mathcal{K}_h$, $\mathrm{Lie}\mathcal{K}_h$ is the maximal subspace on which the action $\mathrm{Ad}_h |\mathrm{Lie}\mathcal{K}$ is I.*

This lemma implies that $\mathrm{Lie}\mathcal{K}/\mathrm{Lie}\mathcal{K}_h$ induces a subspace normal to T_pZ_h in T_pZ.

As in the case of Z/T_z, we can realize Z_h/T_z as a fiber bundle. As in the case of $\mathrm{Lie}\mathcal{K}$, $\mathrm{Lie}\mathcal{K}_h$ splits into $\mathrm{Lie}\mathcal{K}_h'$ and $\mathfrak{t}_p$. And the group $\mathcal{K}_h' \simeq \mathcal{K}_h/T_p^0$ acts effectively on Z_h. Associate with Z_h the space $E_h = Z_h/\mathcal{K}_h'$. Similar to Z, there is the following sequence:

$$Z_h \to Z_h/T_z \to Z_h/\mathcal{K}_h' = E_h$$

where the second projection yields a finite quotient of $\mathcal{K}_h'/T_z$ as the fiber.

3.7. **The action by a Weyl subgroup on the fixed points of T.** Suppose (p, I, z) defines a fixed point of T in X_N, that is to say that $\mathfrak{t}_p, \mathfrak{t}_z$ generate $\mathfrak{t}$, and $\mu(p) = k\tilde{\phi}(z)$.

If $\phi(z)$ is in the interior of C, then $\mathfrak{t}_z = 0$, and $\mathfrak{t}_p = \mathfrak{t}$ by the above characterization of a fixed point.

Suppose $\phi(z) \in \partial C$, let C_ϕ be the smallest wall of ∂C containing ϕ. Then $l\mathfrak{g}_A = (l\mathfrak{g})^{\mathrm{ss}}_\mu$ commutes with $\mu(p) = \phi(z)$, and it is generated by $l\mathfrak{g}_\alpha$ with α vanishing on the wall C_ϕ.

Lemma 3.9. *The subgroup $W^{\mathrm{aff}} \cap (LG)^{\mathrm{ss}}_\mu$ of W^{aff} is the Weyl group of the finite dimensional semi-simple group $(LG)^{\mathrm{ss}}_\mu$. It transforms one fixed point set to another in $\mu^{-1}(k\tilde{\phi})$ where $\tilde{\phi} = \mu(p)/k$.*

For $w \in W(\mathcal{K}')$, it preserves Z but permutes among $\{Z_h\}$.

Pf: The group $(LG)_\mu$ is compact and connected, since it is the stabilizer of $\mu(p) = (\phi(z), k)$ under the adjoint action. The semi-simple part is generated by $(l\mathfrak{g})_\alpha$ where α vanishes on C_ϕ, therefore the Weyl group of $(LG)^{\mathrm{ss}}_\mu$ is generated by the reflections with respect to those affine roots vanishing at C_ϕ. In terms of the original affine Weyl group, it is exactly $W_\mu = W^{\mathrm{aff}} \cap (LG)^{\mathrm{ss}}_\mu$.

Suppose that (p, I, z) defines a fixed point, then $\mu(gp) = \widetilde{\mathrm{Ad}}_g \mu(p) = \mu(p)$, for all $g \in (LG)^{\mathrm{ss}}_p$. If g is also in W_μ, one has $g(\mathfrak{t}_z) = \mathfrak{t}_z$ because $\mathfrak{t}_z$ is the Cartan subalgebra of $(l\mathfrak{g})^{\mathrm{ss}}_\mu$. Thus $g(\mathfrak{t}_p)$ and $g(\mathfrak{t}_z)$ generate $\mathfrak{t}$. To check that (gp, I, z) is fixed by T, let $t \in \mathfrak{t}$,

$$t = t_1 + t_2, \quad t_1 \in g(\mathfrak{t}_p),\, t_2 \in g(\mathfrak{t}_z) = \mathfrak{t}_z.$$

Then

$$\begin{aligned} \exp(2\pi i t)(gp, I, z) &= (gp, I, \exp(2\pi i t) z) \\ &\simeq (\exp(2\pi i t_1) gp, I, \exp(2\pi i t_2) z) \\ &= (g \exp(2\pi i \,\mathrm{Ad}_{g^{-1}} t_1) p, I, z) \\ &= (gp, I, z). \end{aligned}$$

In the above, we have used the fact that $\exp(2\pi i \,\mathrm{Ad}_{g^{-1}} t_1) \in T_p$ if $t_1 \in g(\mathfrak{t}_p)$. This shows the Weyl group of the semi-simple part of $\mu(p)$ acts on the fixed point set whose image is $\mu(p)$.

Fix a lifting of w to $\mathcal{K}$, it certainly preserves Z, since the whole $\mathcal{K}$ does. If Z_h is a stratum as described earlier, associated with $h \in I_p/I_p^0$, for some h and p, then the point $w(p)$ has $\mathrm{Ad}_w T_p$ as stabilizer, and its isotropy group is $\mathrm{Ad}_w I_p$. Since T_z is the Cartan subgroup of $(LG)^{\mathrm{ss}}_\mu$, and w is in its Weyl group, hence w preserves T_z and

$$\mathrm{Ad}_w I_p = I_{w(p)}, \quad \mathrm{Ad}_w I_p^0 = I^0_{w(p)}.$$

Thus $w(Z_h) = Z_{w(h)}$. QED

4. Normal bundles to the fixed point sets in X_N and weights

We will find out in this section the weights of the T-action on the normal space to the fixed point sets inside the compactification locus.

The last section describes the fixed point sets of X_N in terms of data from X and $\mathbb{X}$. Suppose (p, I, z) defines a fixed point in X_N, and (p_s, h_s, z_s) is a curve with

$$(p_0, h_0, z_0) = (p, I, z), \quad (p_0', h_0', z_0') = (a, \xi, x) \in T_{(p,h,z)}(X \times LG \times X_{\mathfrak{g}}).$$

Let $t = t_p t_z$ so that $t_p \in T_p$ and $t_z \in T_z$. Then the action by T on the tangent vectors is given by

$$t(a, \xi, x) = (a, \xi, t(x)) \simeq (t_p(a), \mathrm{Ad}_{t_p} \xi, t_z(x)) \tag{4.1}$$

which is obtained by the definitions of the action of T on X_N, and the equivalence relation as in Def. 3.1.

By assumption, the equivalent class $[p, I, z]$ defines a fixed point in X_N. As before, T_p^0 is the connected component of I in T_p. Then T_p^0 and T_z generate T. Furthermore T_p and T_z act on the tangent spaces T_pX and $T_z\mathbb{X}$, respectively.

4.1. **Tangent space.** First let us describe the tangent space to X_N at $[p, I, z]$.

Proposition 4.1. *Let $\mu = \mu(p)$ and V_p be defined as*

$$V_p = \{a \in T_pX | \exists b \in l\mathfrak{g}, D_{(a+iJa)}\mu(p) = D_{(b+iJb)}\mu_{\mathbb{X}}\}$$

where $\mu_{\mathbb{X}} : \mathbb{X} = LG \times_T X_{\mathfrak{g}} \to l\mathfrak{g} \times \{k\}$. In the toric variety $X_{\mathfrak{g}}$, the point z has T_z as its stabilizer, let the subspace tangent to $T^{\mathbb{C}}(z) \simeq (\mathfrak{t}_z^{\perp})^{\mathbb{C}}$ be denoted by H_z, then the tangent space to X_N at $[p, I, z]$ admits the following decomposition:

$$T_uX_N \simeq V_p \oplus (l\mathfrak{g})_\mu/\mathfrak{t} \oplus H_z$$

where $(l\mathfrak{g})_\mu$ is the Lie algebra of the stabilizer of the μ under the coadjoint action.

The tangent space T_uX_N has a natural almost complex structure J satisfying:

$$J|V_p \simeq J^X; \quad J|(l\mathfrak{g})/\mathfrak{t} = J' = -J^{l\mathfrak{g}}; J|H_z = -J^{X_{\mathfrak{g}}}$$

where $J^X, J^{l\mathfrak{g}}, J^{X_{\mathfrak{g}}}$ are the almost complex structures on the space $X, l\mathfrak{g}/\mathfrak{t}, X_{\mathfrak{g}}$ respectively.

Remark: The map $D\mu$ after extended to the complexified tangent space, certainly is not holomorphic, thus the vectors in V_p are very special.

Pf: The tangent space of T_uX_N is isomorphic to the space

$$\{(a, b) \in T_{(p,q)}X \times \mathbb{X} | D_{(a+iJa)}\mu(p) = D_{(b+iJ'b)}\mu_{\mathbb{X}}(q)\},$$

as shown in Section 2 (or cf. [C1]). The subspace H_z is contained there via $b \mapsto (0, 0, b)$, for $D_{(0,b+iJ'b)}\mu_{\mathbb{X}}(I, z) = 0$. Let $q = [I, z] \in \mathbb{X}$. The subspace $l\mathfrak{g}_\mu/\mathfrak{t}$ is contained there via the inclusion $\xi \in l\mathfrak{g}_\mu/\mathfrak{t} \mapsto (0, \xi, 0)$, and it satisfies the equation

$$D_{(\xi+iJ'\xi,0)}\mu_{\mathbb{X}}(I, z) = D_{(\xi-i\xi^J,0)}\mu_{\mathbb{X}}(I, z) = [\xi - i\xi^J, \mu_{\mathbb{X}}(I, z)] = 0,$$

since $[\xi, \mu_{\mathbb{X}}(q)] = [\xi^J, \mu_{\mathbb{X}}(q)] = 0$. Also we have used the fact that $J'\xi(q) = -J^{l\mathfrak{g}}\xi(q) = -\xi^J(q)$ with $J^{l\mathfrak{g}}\xi = \xi^J \in l\mathfrak{g}_\mu/\mathfrak{t}$ being defined by the almost complex structure on $l\mathfrak{g}_\mu/\mathfrak{t}$.

The subspace defined by $(l\mathfrak{g})_\mu/\mathfrak{t} \oplus H_z$ corresponds exactly to the subspace

$$U_0 = \{(a, b) \in T_{(p,q)}X \times \mathbb{X} | a = 0\}.$$

The complement of U_0 in T_uX_N is exactly V_p.

There is a unique way to choose b in the above definition of V_p: make it perpendicular to H_z, and choose it from

$$\sum_{\alpha \notin \Delta(l\mathfrak{g}_\mu)} l\mathfrak{g}_\alpha$$

where the index means α is not a root of $l\mathfrak{g}_\mu$. This is possible since $D_b\mu_{\mathbb{X}} = D_{J'b}\mu_{\mathbb{X}} = 0$ if $b \in H_z$ or b is induced from $l\mathfrak{g}_\mu$.

Then the map $h : a \mapsto (a, b)$ satisfies $h \cdot J = J' \cdot h$. So the complex structure defined by J^X on V_p is mapped to $J = (J^X, J')$ restricted to the image of h. Thus the assertion is verified. The other two idenities involving J' follow strictly from the description of J' on $T\mathbb{X}$. QED

4.2. **Normal space to Z and Z_h.** From the description of V_p and $l\mathfrak{g}_\mu/\mathfrak{t}$, it is clear they inherit an almost complex structure from $T_{[p,I,z]}X_N$. In the following the weights refer to the weights by $\mathfrak{t}_p, h \in I_p \setminus I_p^0$ acting on the complex linear spaces.

Let V_p^0 and N_p be the 0-weight and non-zero weight subspaces of V_p under the linear t_p-action respectively. Then $V_p = N_p \oplus V_p^0$.

Similarly, denote by $\mathrm{nor}^1(Z_h, Z) \subset \{V_p^0\}$ the maximal h-stable subbundle subspace on which $\det(I - h) \neq 0$.

Proposition 4.2. *1). The normal subspace to the fixed point set Z/T_Z is*

$$N_p \oplus (l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K} \oplus H_z.$$

2). The normal bundle of Z_h in Z can be decomposed as

$$\mathrm{nor}^1(Z_h, Z) \oplus \mathrm{Lie}\mathcal{K}/\mathrm{Lie}\mathcal{K}_h.$$

The bundle $\mathrm{nor}^1(Z_h, Z)$ admits an action by T_z which lifts the action on Z_h.

On the normal subspaces in both situation, there is the induced almost complex structure from J on T_uX_N.

Pf: The first claim holds due to the fact that the normal space to the fixed point set is the non-zero weight space under the action by the $\mathfrak{t}_p$ on the tangent space. From the description of the tangent space and the definition of N_p, the assertion is evident.

Part 2 follows from essentially the same reasoning, with a slight variation since here we consider only the h-action. The the normal space is the maximal h-stable subspace in T_pZ, on which 1 is not an eigenvalue. There are two factors in the tangent to T_pZ, one is the $\mathrm{Lie}\mathcal{K}/\mathfrak{t}$, the other is V_p^0. Then from the definition of $\mathrm{nor}^1(Z_h, Z)$ and Lemma 3.8, the decomposition holds.

The action by T_z on $\mathrm{nor}^1(Z_h, Z)$ exists since T_z commutes with h, so the non-zero weight space under h in V_p at p is isomorphic to that in V_{tp}, $t \in T_z$.

The almost complex structure preserves those subspaces since $\mathfrak{t}_p$ and h-actions commute with J. QED

4.3. **Description of weights.** From the last section, it is verified that $\mathfrak{t} = \mathfrak{t}_p \oplus \mathfrak{t}_z$ and $\mathfrak{t}_p \cap \mathfrak{t}_z = 0$. Let $t = t_p + t_z$, $t \in \mathfrak{t}$ denote this decomposition.

Proposition 4.3. *Let the weights of the T_p-action on N_p be denoted by $\{\gamma\}$, the weights of the T_z-action on H_z denoted by $\{\lambda\}$, and the positive roots of $(l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}$ be denoted by $\{\beta\}$ respectively.*

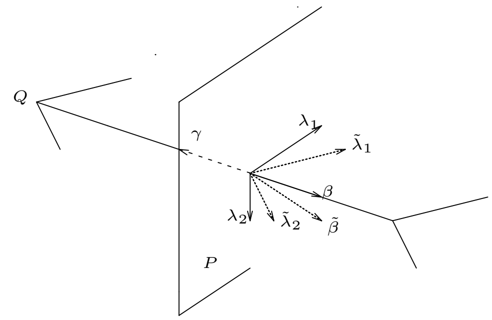

FIGURE 4.1. Weights at the fixed point set

Then the weights by the T-action on $T_{[p,I,z]}X_N$ are given by the corresponding three sets of weights: $\{\tilde{\gamma}\}$, $\{\tilde{\lambda}\}$, $\{\tilde{\beta}\}$ such that

$$\tilde{\gamma}(t) = \gamma(t_p), \quad \tilde{\lambda}(t) = -\lambda(t_z), \quad \tilde{\beta}(t) = -\beta(t_p). \tag{4.2}$$

Remark: The negative sign for $\tilde{\lambda}$ and $\tilde{\beta}$ reflects the complex structure on X_N as described in Prop. 4.1.

Pf: The action by t on the tangent space $T_{[p,I,z]}X_N$ is of the form:

$$t_p \cdot t_z(a, \xi, x) = (t_p(a), [t_p, \xi], t_z(x))$$

as shown by Eq. (4.1). Now apply this to the three types of normal vectors as in the last proposition, one has the assertion. The only thinking needed here is the observation that the embedding of V_p into T_uX_N is T_p equivariant. The '-' sign reflects the choice of the complex structure on $H_z, l\mathfrak{g}_\mu/\mathfrak{t}$. QED

4.4. **Weights on normal space to the lower strata Z_h in the fixed point set.** As pointed earlier, the normal space to Z_h consists of two parts: the normal space to Z and the normal space of Z_h in Z. The contribution of Z_h to the fixed point formula, in the orbifold sense, depends on the weights of the action by hT_p^0 on the normal space. The calculation given earlier provides the answer for the action by T on the normal space to Z, here we need only determine the weights on the second subspace which is $\text{nor}^1{}_p(Z_h, Z) \oplus (\text{Lie}\mathcal{K}/\text{Lie}\mathcal{K}_h)$. Recall that $h \in I_p/I_p^0 = (T_p \cap T_z)/(T_p^0 \cap T_z)$.

Lemma 4.1. *Suppose the weights of the action by h on $\text{nor}^1(Z_h, Z)$, $(\text{Lie}\mathcal{K}/\text{Lie}\mathcal{K}_h)$ are given by $\{\theta_i\}$, $\{\beta_i\}$ respectively. Let $(ht_p, h^{-1}t_z) \in hT_p^0 \times T_z$ be a lifting of $t \in T$ in $T_p \times T_z$, $\{\tilde{\theta}\}, \{\tilde{\beta}\}$ be the weights of the action by $(ht_p, h^{-1}t_z)$ on the two factors of the normal space. They are given by*

$$\tilde{\theta}(ht_p \cdot h^{-1}t_z) = \theta(h), \quad \tilde{\beta}(ht_p \cdot h^{-1}t_z) = \beta(h).$$

Pf: Let (p_s, k_s, z) be a curve, with p_s, k_s tangent to $Z, \mathcal{K}'$ but normal to $Z_h, \mathcal{K}'_h$ and $p_0 \in Z_h, k_0 \in \mathcal{K}'_h$; and z is fixed by T_z. The action by $t \in T$ on that curve after choosing the lifting $ht_p \cdot h^{-1}t_z$, is simply

$$(ht_p p_s, \text{Ad}_{ht_p} k_s, z),$$

which is $(hp_s, \text{Ad}_h k_s, z)$ since t_p acts trivially on Z, and $\text{Ad}_{t_p} k = k, \forall k \in \mathcal{K}$ by the definition of $\mathcal{K}$. Therefore, the weights are of the forms described. QED

4.5. **Three types of fixed points.** Suppose (p, I, z) defines a fixed point in X_N, we classify them according to the following:

1). $\mu(p)/k = \tilde{\phi}(z)$ is in the interior of $(C, 1)$.

2). $\mu(p)/k = \tilde{\phi}(z)$ is on $(\partial C, 1)$ but not a vertex of C.

3). $\mu(p)/k = \tilde{\phi}(z)$ is one of the vertices of the simplex $(C, 1)$.

Considering the decomposition of $\mathfrak{t}$ into $\mathfrak{t}_p, \mathfrak{t}_z$, the type 1) and 3) correspond to the cases $\mathfrak{t}_z = 0$ and $\mathfrak{t}_p = 0$ respectively.

4.6. **The $(LG)_\mu/T$ factor.** In case of type 3), $\mathfrak{t}_z = \mathfrak{t}$, therefore $\mathfrak{t}_p = 0$. Hence $\mathcal{K} = (LG)_\mu$, and $(LG)_\mu$ acts on the fixed point set, as shown for $\mathcal{K}$. What's said earlier about the map $\pi : F \to E$ with fiber $\mathcal{K}'/T_z$ now can be replaced by the factor $(LG)_\mu/T$.

4.7. **More on W^{aff}.** Suppose the fixed point is of either type 2) or 3), then $\mu(p)$ has stabilizer $(LG)_\mu$, under the co-adjoint action. The group has semi-simple part generated by $(l\mathfrak{g})_\alpha$, where α is an affine root vanishing at $\mu(p)/k$.

The following is a well known fact in affine algebra:

Lemma 4.2. *The group generated by the reflections*

$$r_\alpha = \{\alpha | \alpha(\mu) = 0\}$$

is the Weyl group of T in $(LG)_\mu$, it acts on the type 2) fixed point set. Denote the Weyl group W_μ which is the subgroup of W^{aff} fixing the smallest wall containing μ.

4.8. **Transformation weights by W_μ.** The discussion here is only meaningful for type 2) and 3) fixed points.

In the previous section, it was shown that the Weyl group W_μ permutes among the collection of fixed point sets with the same value for μ.

Let $\{\tilde{\lambda}'\}, \{\tilde{\beta}'\}, \{\tilde{\gamma}'\}$ denote the three groups of weights at the point (wp, I, z) as in Prop. 4.3.

Using the expressions in Eq. (4.1) and Prop. 4.3, we conclude that all the non-trivial weights are those $\tilde{\lambda}$'s in case of type 3) fixed point, since $\mathfrak{t}_p = 0$ and $\mathcal{K} = LG_\mu$, so there is no γ, β. Hence, along the $(LG)_\mu$-orbit of (p, I, z) which is fixed by T in X_N, the weights are just those from H_z, $\{\lambda\}$.

In case of type 2) fixed point, we have the following relation between the stabilizers:

$$T_{w(p)} = wT_p.$$

To derive the transformation rule for the weights, we need the following:

Lemma 4.3.

$$V_{w(p)} = w_*(V_p).$$

where $w_* : T_pX \to T_{w(p)}$ is the isomorphism induced from the diffeomorphism w.

Pf: Let $a \in V_p$, i.e. $\exists \xi \in l\mathfrak{g}/\mathfrak{t}$ such that $D_{a+iJa}\mu(p) = D_{\xi+iJ'\xi}k\tilde{\phi}(z)$. Then using the invariance of J under w, we have

$$\begin{aligned} D_{w(a+iJa)}\mu(w(p)) &= w(D_{a+iJa}\mu(p)) \\ &= w(D_{\xi+iJ'\xi}k\tilde{\phi}) \\ &= D_{w(\xi)+iJ'w(\xi))}\mu, \end{aligned} \tag{4.3}$$

since w preserves $\mathfrak{t}$, we have $w(\xi) \in l\mathfrak{g}/\mathfrak{t}$. So the assertion holds. QED

Write t in two different way:

$$t = t_p + t_z \in \mathfrak{t}_p \oplus \mathfrak{t}_z; \quad t = t_{wp} + t'_z \in \mathfrak{t}_{wp} \oplus \mathfrak{t}_z,$$

how are t_p, t_{wp} related?

Lemma 4.4. $wt_p = t_{wp} \in t_{wp}, \quad t'_z == t - wt + wt_z = t_p - wt_p + t_z \in \mathfrak{t}_z.$

Pf: From the first equation, we have $wt = wt_p + wt_z$. Also $W_\mu = W(l\mathfrak{g}_m u)$ and $\mathfrak{t}_z = l\mathfrak{g}_\mu^{\text{ss}} \cap \mathfrak{t}$, we claim $wt - t \in t_z, \forall w \in W_\mu$. The claim holds easily for reflections, hence it holds for all w. Thus

$$t = wt + (t - wt) = wt_p + (t - wt + wt_z) \in \mathfrak{t}_{wp} \oplus \mathfrak{t}_z.$$

Hence $wt_p = t_{wp}$, $t'_z = t - wt + wt_z$ by uniqueness of decomposition. Another expression for t'_z is

$$t'_z = t - wt + wt_z = (t_p + t_z) - w(t_p + t_z) + wt_z = t_p - wt_p + t_z. \quad \text{QED}$$

Let $\{\gamma'\}$ be the weights of the $T_{w(p)}$-action on $V_{w(p)}$, then one has

$$\gamma' = w(\gamma).$$

Associated with γ' is a weight $\tilde{\gamma}'$ which is a character of $T^0_{wp} \times T_z$, as shown in Prop. 4.3, defined by $\tilde{\gamma}'(t) = \gamma'(t_{w(p)})$. Hence

$$\tilde{\gamma}'(t) = \gamma'(t_{w(p)}) = w(\gamma)(wt_p) = \gamma(t_p) = \tilde{\gamma}(t). \tag{4.4}$$

The first group of weights on $T_{(p,I,z)}X_N$, denoted by $\{\tilde{\gamma}\}$ as in Prop. 4.3 are invariant under W_μ, see Fig 4.1. The group W_μ are generated by reflections w.r.t. planes containing γ.

Those weights $\{\tilde{\beta}'\}$ are defined by $\tilde{\beta}'(t) = -\beta(t_{w(p)})$. But $t_{w(p)} = w(t_p)$, therefore

$$\tilde{\beta}'(t) = -\beta(t_{w(p)}) = -w(\beta)(t_p) = w(\tilde{\beta})(t). \tag{4.5}$$

The weights $\{\tilde{\lambda}'\}$, are defined as

$$\tilde{\lambda}'(t) = -\lambda(t'_z). \tag{4.6}$$

The following transformation law for the weights is essential for future calculations.

Proposition 4.4. *Let $\Pi : \mathfrak{t} \to \mathfrak{t}_z$ be the orthogonal projection, then*

$$\tilde{\lambda}(t) = \lambda(\Pi t) - \lambda(\Pi t_p), \quad \tilde{\beta}(t) = \beta(\Pi t_p). \tag{4.7}$$

At the point (wp, I, z) with $w \in W_\mu$, the weights are given by

$$\tilde{\gamma}' = \tilde{\gamma}, \quad \tilde{\lambda}'(t) = \lambda(\Pi t) - \lambda(w\Pi t_p), \quad \tilde{\beta}'(t) = \beta(w\Pi t_p). \tag{4.8}$$

For $v \in W_\mu$,

$$\begin{aligned} \tilde{\gamma}'(vt) &= \tilde{\gamma}'(t) = \tilde{\gamma}(t), \\ \tilde{\lambda}'(vt) &= v\lambda(\Pi t) - w\lambda(\Pi t_p), \\ \tilde{\beta}'(vt) &= w\beta(\Pi t_p). \end{aligned} \tag{4.9}$$

Remark: 1). The projection Π can be removed since β, λ are linear functions on $\mathfrak{t}_z$ and their extension to $\mathfrak{t}$ by convention are compositions with Π. 2). If $w \in W(\text{Lie}\mathcal{K}')$, then $wt_p = t_p$ since $[\text{Lie}\mathcal{K}', \mathfrak{t}_p] = 0$, and $\tilde{\beta}' = \tilde{\beta}$.

Pf: The proof is divided into 3 steps for each group of equations in the above.

1). Since $t_z \in \mathfrak{t}_z$, and $t = t_z + t_p$, apply the map Π, one has $t_z = \Pi t - \Pi t_p$. Hence

$$\tilde{\lambda}(t) = \lambda(t_z) = \lambda(\Pi t) - \lambda(\Pi t_p).$$

The root β vanishes on the orthogonal complement of $\mathfrak{t}_z$, thus

$$\tilde{\beta}(t) = \beta(t_p) = \beta(\Pi t_p).$$

2). An useful observation can be made here:

$$\Pi(wt) = w\Pi(t) \tag{4.10}$$

which is true for w given by reflection r_β with respect to a root β of $(LG)_\mu$, for

$$\Pi(r_\beta t) = \Pi(t_\beta - \beta(t)\beta^{\mathrm{v}}) = \Pi(t_\beta) - \beta(t)\beta^{\mathrm{v}}$$

Since $\Pi(\beta^{\mathrm{v}}) = \beta^{\mathrm{v}}$, also $\beta(t) = \beta(\Pi t)$. Combine the two one gets

$$\Pi(r_\beta t) = r_\beta(\Pi t).$$

Hence it holds for any $w \in W_\mu$.

It also has been shown that $\tilde{\gamma}' = \tilde{\gamma}$. As for $\tilde{\lambda}'$ and $\tilde{\beta}'$, using $t'_z = t - t_{wp} = t - wt_p$, we obtain

$$\begin{aligned}\tilde{\lambda}'(t) &= \lambda(t'_z) = \lambda(\Pi t) - \lambda(\Pi t_{w(p)}),\\ \tilde{\beta}'(t) &= \beta(t_{w(p)}) = \beta(wt_p) = w\beta(t_p) = w\beta(\Pi t_p).\end{aligned}$$

3). Decompose $t = t_p + t_z = v^{-1}t_p + t_z + (t_p - v^{-1}t_p)$, we've shown that $t_p - v^{-1}t \in \mathfrak{t}_z$, since $v \in W_\mu$. Therefore $t'_z = t_z + (t_p - v^{-1}t) \in \mathfrak{t}_z$ and $v(t) = t_p + v(t'_z)$ with $v(t'_z) \in \mathfrak{t}_z$, thus $(vt)_p = t_p$ and

$$\tilde{\gamma}'(vt) = \tilde{\gamma}(vt) = \gamma((vt)_p) = \gamma(t_p) = \tilde{\gamma}(t).$$

As shown in Step 2), by replacing t there with vt,

$$\tilde{\lambda}'(vt) = \lambda(\Pi vt) - \lambda(w\Pi(vt)_p),$$

but $\Pi vt = v\Pi t$, and it has just been shown $(vt)_p = t_p$, so we have

$$\begin{aligned}\tilde{\lambda}'(vt) &= \lambda(v\Pi t) - \lambda(w\Pi t_p)\\ &= v\lambda(\Pi t) - w\lambda(\Pi t_p).\end{aligned} \tag{4.11}$$

For $\tilde{\beta}'$, following Step 2), one has

$$\tilde{\beta}'(vt) = w\beta(\Pi(vt)_p),$$

but $(vt)_p = t_p$, therefore

$$\tilde{\beta}'(vt) = w\beta(\Pi t_p). \quad \text{QED}$$

Thus we have found how the weighs are related along W_μ-orbit, and how they change under $t \mapsto vt$.

4.9. **Transformation of weights for Z_h.** For the $p \in Z_h$, the group $W(\mathcal{K})/W(\mathcal{K}_h)$ acts on it, in addition to the transformation by w in $W((LG)_\mu)/W(\mathcal{K})$. The first group preserve Z but permutes among $\{Z_h\}$. Since both $(LG)_\mu$ and $\mathcal{K}$ contain T, there is the obvious exact sequence

$$W(\mathcal{K})/W(\mathcal{K}_h) \to W((LG)_\mu)/W(\mathcal{K}_h) \to W((LG)_\mu)/W(\mathcal{K}).$$

The strata $Z_{vh}/T_z \times \{z\}$ contains $[vp, I, z]$, where $Z_{vh} = vZ_h$. The normal space to $Z_{vh}/T_z \times \{z\}$ acquires two more set of weights $\{\theta_i^v\}$ and $\{\beta^v\}$ as shown earlier.

Let $t \in T$, with a fixed decomposition $t_z t_p$, the action on the normal space at $[vp, I, z]$ is that of $vht_p \cdot (vh)^{-1} t_z$.

The assertions in the next lemma are already verified in Lemma 4.1 and Prop. 4.4

Lemma 4.5. *Let $v \in W(\mathcal{K})/W(\mathcal{K}_h)$. The weights $\{\tilde{\theta}_i^v\}$ and $\{\tilde{\beta}^v\}$ satisfy the following:*

$$\tilde{\theta}^v(vht_p \cdot (vh)^{-1}t_z) = \theta(h), \quad \tilde{\beta}^v(vht_p \cdot (vh)^{-1}t_z) = \beta(vh).$$

If $v \in W((LG)_\mu)/W(\mathcal{K})$, v does not preserves T_p^0. The lifting of t is given by $(vh)t_{vp} \cdot (vh)^{-1}t'_z$, where $t_{vp} = vt_p$. With that lifting, the weights $\{\tilde{\theta}^v\}, \{\tilde{\beta}^v\}$ are given by the same formula.

In both cases, the weights of the action by $((vh)t_{vp}, (vh)^{-1}t'_z)$ in the normal space of vZ in vF are simply given by the evaluations of the weights given in Prop. 4.4 on $((vh)t_{vp}, (vh)^{-1}t'_z)$

4.10. **A word about lifting of action by t.** For orbifolds, in order to evaluate the contribution by the fixed point set, it is necessary to consider all the lifting of the action at a fixed point to the finite smooth cover. Once there, one needs to find the fixed point set of each lifting and find its contribution. The strata Z_h/T_z for $h \in I_p \setminus I_p^0$ is one of those fixed point sets. What if $h \in I_p^0$? For such a h, $(ht_p, h^{-1}t_z) \in T_p \times T_z$ is a lifting of $t \in T$ as well, but $ht_p \in T_p^0$, and hence the consideration for that lifting is already incorporated when we study the action by general $(t_p, t_z) \in T_p^0 \times T_z$.

5. Curvatures of various bundles

In order to understand the contribution from the fixed points coming from compactification, i.e. those with images on $W(\partial C)$, we need to know the curvature of their normal bundles. And their transformation law by certain subgroups of W^{aff}.

5.1. **A general fact.** Suppose S acts on a manifold N and $\mathbb{C}^n$, the action on N is locally free and the action on $\mathbb{C}^n$ is linear defined by $\lambda : \mathrm{Lie}S \to \mathfrak{g}l(\mathbb{C}^n)$. Let A be a S-invariant connection on N with $\mathrm{Lie}S$ identified with the vertical subspace in TN. Denote the curvature of the connection by F_A which is a $\mathrm{Lie}S$-values horizontal two form.

Proposition 5.1. *1). Then $(N \times \mathbb{C}^n)/S$ as vector bundle over N/S has a connection whose curvature is given by $\lambda(F_A)$.*

2). Suppose $V = \mathbb{C}^n/\Lambda$ where Λ is a finite group acting on the vector space linearly. If S acts on $N \times V$, such that the action is locally free on N, and is a linear action on the orbifold line bundle V, i.e. there is an extension of S by the finite group Λ, S' acting on $\mathbb{C}^n$ linearly. Then the orbifold line bundle $(N \times V)/S$ over N/S has the same curvature form as in 1).

Pf: 1). On the trivial bundle $N \times \mathbb{C}^n$, defines the following connection $d + \lambda \cdot A$, notice that $\lambda \cdot A : TN \to \mathfrak{g}l(\mathbb{C}^n)$ is indeed a 1-form. This connection has the feature that the curve $(g_t(p), g_t(v))$ is horizontal for any 1-parameter subgroup $\{g_t\}$. Therefore, it descends to a connection on the quotient space N/S. For the quotient connection constructed this way, the curvature is given by the descent of the curvature upstairs which is of the given form.

2). Let S' act on N through S. Then apply the above argument to the extension group S'. QED

5.2. **Action by T_z on bundles.** The group T_z has a local free action on either Z or Z_h. The action is obtained by restricting its action on X to the given sets. Therefore it acts on the various subbundles of the normal bundles. Denote the action on normal bundle by $dt_z, \forall t_z \in T_z$.

Lemma 5.1. *1). $dt_z : V_p \to V_{t_z p}$ is an isomorphism preserving the decomposition into $V^0 \oplus N$.*

The map dt_z extends to an isomorphism

$$dt_z : N_p \oplus (l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K} \oplus H_z \to N_{t_z p} \oplus (l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K} \oplus H_z. \tag{5.1}$$

2). The quotient by T_z as described in Eq. (5.1) defines the normal orbifold line bundle to the fixed point set F in X_N.

Warning: The action by t_z above should not be confused with the T-action on normal bundle to the fixed points studied in Section 3. The action here defines the equivalent class, while the action in Section 3 is on the set of equivalent classes.

Pf: Let (p_s, h_s, z_s) be a curve with $(p_0, h_0, z_0) = (p, I, z)$. Let $q = [I, z] \in \mathbb{X}$. Using the defining equivalence relation, the diagonal action by T_z on the first two factors is

$$(p_s, h_s, z_s) \mapsto (t_z p_s, t_z h_s, z_s) \simeq (t_z p_s, t_z h_s t_z^{-1}, t_z^{-1} z_s).$$

Differentiate the above at $s = 0$ to obtain the diagonal action by t_z on the tangent vectors:

$$dt_z(p', h', z') \simeq (t_{z*}(p'), Ad_{t_z} h', t_z^{-1}(z')). \tag{5.2}$$

If $(p', h', z') \in V_p$, we have $z' = 0$ and

$$D_{p'+iJp'}\mu(p) = D_{h'+iJ'h'}k\mu_{\mathbb{X}}(q). \tag{5.3}$$

Let t_{z*} denote the induced action by t_z on the tangent space. Apply the action Ad_{t_z} on both sides, and combine that with the properties of $d\mu$, one has

$$\begin{aligned} D_{t_{z*}(p'+iJp')}\mu(t_z p) &= \mathrm{Ad}_{t_z}(D_{(p'+iJp')}\mu(p)) \\ &= \mathrm{Ad}_{t_z}(D_{(h'+iJh')}\mu_{\mathbb{X}}(q)) \\ &= D_{t_{z*}(h'+iJh')}\mu_{\mathbb{X}}(q), \end{aligned}$$

which shows that $(t_{z*}(p'), t_{z*}(h'), 0) \in V_{t_z p}$.

The calculation above also shows for general $g \in LG$, the following holds:

$$g_*(p' + iJp', h' + iJh', 0) = (g_*(p' + iJp'), \mathrm{Ad}_g(h' + iJh'), 0). \tag{5.4}$$

Since T_p, T_z commute, one has $T_{t_z p} = T_p$, and clearly $(p' + iJp', h' + iJh', 0)$ has a non-zero weight under T_p iff $(t_{z*}(p' + iJp'), t_{z*}(h' + iJh'), 0)$ has non-zero weight under $T_{t_z p} = T_p$. Hence N_p is $N_{t_z p}$ under t_{z*}.

The map t_{z*} preserves $(l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K}$ since $\mathfrak{t}_z$ is the Cartan subalgebra of the semi-simple part of $(l\mathfrak{g})_{\tilde{\phi}}$, and T_z preserves $\mathrm{Lie}\mathcal{K}$ under the adjoint action. The map t_{z*} preserves H_z because T_z fixes z and H_z is the subspace in $T_z X_{\mathfrak{g}}$ with non-zero weights. So it is stable under the action by T_z.

2). The normal subspace to the fixed point set consists of non-zero weight vectors in the tangent space $V_p \oplus (l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K} \oplus H_z$. Therefore, it is given by $N_p \oplus (l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K} \oplus H_z$ before mod out by the local free action of T_z. QED

5.3. **Curvatures of the normal bundles.** Next we calculate the curvatures of the three subbundles given in the above. The transformation of the curvatures under the action by W_μ will be given as well.

Following Eq. (5.2), for (p', h', z') in $(l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K}$, i.e., $p' = 0, z' = 0$, and $h' \in (l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K}$, the action by T_z is the adjoint action. And the action by T_z on H_z is the action by isotropy group, since T_z fixes z.

Based on that Eq. (5.2), one can write the curvature in terms of the weights and the connection A as follows:

Proposition 5.2. *1).The curvature of the bundles*

$$Z \times_{T_z} ((l\mathfrak{g})_{\tilde{\phi}}/\mathrm{Lie}\mathcal{K}), \quad Z \times_{T_z} H_z$$

are given by

$$-\oplus_{\beta \in \Delta_\mu/\Delta(\mathrm{Lie}\mathcal{K})} \beta \cdot dA, \quad -\oplus_\lambda \lambda \cdot dA$$

respectively.

2). Let $\widetilde{\nabla}$ be a T_z-invariant connection on $\tilde{N} = \{N_p\}_{p \in Z}$ over $\tilde{F}_p \cap \mu^{-1}(\phi)$, and $a_q(t) = \mathcal{L}_t - \widetilde{\nabla}_t$, where $\mathcal{L}_t$ is the Lie derivative of t acting on the bundle $\tilde{N}$, then the curvature of N is given by

$$(\widetilde{\nabla})^2 + a_q(dA).$$

Remark: A simple but potentially important observation is that t_z acts in Eq. (5.2) on the last component z_s as t_z^{-1}. This introduces the $-$ sign in the next proposition. Hence the weights are $\{-\lambda\}$ while the curvatures are given by $\{\lambda dA\}$. For the bundle defined by $l\mathfrak{g}_\mu/\mathfrak{t}$, the weights are $\{-\beta\} = -\Delta(l\mathfrak{g}_\mu)_+$ while the curvatures are $-\{\beta dA\}$.

5.4. **Transformations of curvatures under W_μ.** Recall that the fixed point sets $\tilde{F}_p \cap \mu^{-1}(\phi)$ transform under the action by the subgroup in W^{aff}, W_μ, which is the Weyl group of $(LG)_\mu$. We want to know first how the transformations act on the normal bundles and how the curvatures change.

If the following, for $w \in W_\mu$, we fix a lifting to $(LG)_\mu$ and it will be denoted the same.

It follows directly from Eq. (5.4), at (wp, I, z), the bundle $V_{wp} = w^* V_p$. Also, $T_{wp} = wT_z$, thus $v \in V_p$ is of non-zero weight with respect to T_p iff $w^*(v)$ is of the same weight with respect to the action by $wT_p = T_{wp}$. Therefore, one has $N_{wp} = w^* N_p$. On the other hand, T_z is the Cartan subgroup of $(LG)_\mu^{\mathrm{ss}}$ which is the semi-simple part of $(LG)_\mu$, whence W_μ is the normalizer of T_z. And conveniently we have $\{N_{wp}\}/T_{wp} = w(\{N_p\}/T_p)$. Hence we have proved the first part of the following:

Proposition 5.3. *1). The pull-back to $\{N_p\}$ by w of curvature form $(\widetilde{\nabla})^2 + a_q(dA)$ of the bundle $\{N_{wp}\}$ at wp is the same as that of $\{N_p\}$ at p under the map w^*.*

2). The pull-back under w of the bundles $wZ \times_{T_z} (l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}$, $wZ \times_{T_z} H_z$ have curvatures given respectively by

$$- \oplus_{\beta \in \Delta_\mu/\Delta(\mathrm{Lie}\mathcal{K})} w^{-1}(\beta) \cdot dA, \quad \oplus_\lambda w^{-1}(\lambda) \cdot dA.$$

Pf: The two expressions in Part 2 can be verified the same way. Let's take the first one. For each subalgbera $l\mathfrak{g}_\beta$, one has the isomorphism:

$$w^* : \tilde{F}_p \cap \mu^{-1}(\phi)) \times \mathrm{Ad}_{w^{-1}}(\mathfrak{g}_\beta) \to \tilde{F}_{wp} \cap \mu^{-1}(\phi)) \times \mathfrak{g}_\beta.$$

The action by T_z on $\mathrm{Ad}_{w^{-1}}(\mathfrak{g}_\beta)$ has weight $-w^{-1}(\beta)$. Therefore the curvature is given by $-w^{-1}(\beta) \cdot dA$. QED

5.5. **More on the curvature of the bundle N.** Recall $\tilde{N} = \{N_p\}_{p \in Z}$, and N is the quotient $\tilde{N}/T_z$. The group $\mathcal{K}'$ acts on $\tilde{N}$ in a obvious manner, since $\mathcal{K}'$ commutes with T_p. Let $\widetilde{\nabla}$ be a $\mathcal{K}'$-invariant connection on the bundle $\tilde{N}$, and the associated moment map ϵ be defined as

$$< \epsilon(q), (\xi) > := \mathcal{L}_\xi - \widetilde{\nabla}_\xi \in \hom(N_q).$$

Since we have $PA : TZ \to \mathrm{Lie}\mathcal{K}'$, we can change ∇ to a new connection by adding a 1-form

$$\nabla = \widetilde{\nabla} + < \epsilon(q), PA(\cdot) > .$$

The invariant sections of $\tilde{N}$ now are horizontal with respect to ∇.

Using invariant sections on $\tilde{N}$, one proves easily the following:

Proposition 5.4. *The connection ∇ descends respectively to connections on $N \to F$ and $\tilde{N}/\mathcal{K}' \to Z/\mathcal{K}' = E$. The curvature 2-form on F is the pull-back of that over E. In particular the curvature is trivial on the fiber $\pi : F \to E$.*

5.6. **The curvature form of bundles over Z_h/T_z.** Along the strata Z_h/T_z, there is the normal bundle in Z: $\mathrm{nor}^1(Z_h, Z) \oplus \mathrm{Lie}\mathcal{K}/\mathrm{Lie}\mathcal{K}_h$ in addition to the restriction to Z_h/T of the normal bundle of $Z/T_z \times \{z\}$ in $TX_N|_{Z_h/T_z \times \{z\}}$.

The curvatures and their transformations under W_μ of those two bundles can be written down similar to what we have just done for Z. Recall that Z_h admits a locally free action by $\mathcal{K}'_h$, and Z_h/T_z is a fiber bundle with fiber given by a finite quotient of $\mathcal{K}'_h/T_z$. On Z_h fix a $\mathcal{K}'_h$-invariant connection A_h, so that the tangent space is decomposed as $T^H Z_h \oplus T^\perp Z_h$, and $P_h : T^\perp Z_h \to Z_h \times \mathrm{Lie}\mathcal{K}'_h$ is the

trivialization of the vertical bundle. The map P_h performs the same function as that of P for Z. Fix a connection for the bundle $T^\perp Z$ which satisfies $\nabla_\xi = \mathcal{L}_\xi, \forall \xi \in \mathrm{Lie}\mathcal{K}'_h$. Its existence follows the discussion on Z with respect to the group $\mathcal{K}'$.

For convenience, we group the two components in the normal space to Z_h, $(l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}$ and $\mathrm{Lie}\mathcal{K}/\mathrm{Lie}\mathcal{K}_h$ together as $(l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}_h$. They received separate treatment earlier since the first one is normal to Z and the second one is tangent to Z but normal to Z_h.

The proof of the following is identical to that of Prop. 5.3, just replace Z_h by Z.

Proposition 5.5. *The curvature of ∇ is the pull-back of a form on $Z_h/\mathcal{K}_h$.*

The curvature of the bundle $Z_h \times_{T_z} ((l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}_h)$ on Z_h/T_z is given by $-\sum_\epsilon \epsilon \cdot dP_h A_h$ where ϵ are the simple roots of $(l\mathfrak{g})_\mu$ not in $\mathrm{Lie}\mathcal{K}_h$. Furthermore, the curvature form $R_h = dP_h A_h$ can be decomposed into

$$dP_h A_h = B_h + R_h$$

where $B_h(u,v)(pg)$ and $R_h(\xi,\eta)$ satisfy the same property as that of B, R in Lemma 3.4.

For $w \in W_\mu$, the normal bundle at $wp \in Z_{wh}$ is given by

$$wV_p^{01} \oplus wN_p \oplus (l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}_{wh} \oplus H_z,$$

where V_p^{01} is the maximal h-stable subspace of V_p^0 on which $\det(I-h) \neq 0$. The curvature of the first two components are the same as those at $p \in Z_h$. The curvature of the last two components are given respectively by

$$-\oplus w^{-1}(\beta) \cdot dP_h A_h, \quad \oplus w^{-1}(\lambda) \cdot dP_h A_h.$$

6. A fundamental formula

In order to prove the desired fixed point formula, we need to understand the contribution from the fixed points in X_N of type 2) and 3). Those do not come as fixed points of X itself under the T-action, rather from certain compactification, X_N, of the quotient X by the nilpotent subgroup $LG^{\mathbb{C}+}$. For in the space X_N, there is an open dense set which corresponds to X/LG^+.

We will prove in this section an important formula which enables us to calculate the contribution from the type 2) and 3) fixed point set in the next section.

Suppose $\mathfrak{k}$ is the Lie algebra of the semi-simple compact Lie group K, with $\mathfrak{t}$ as its Cartan subalgebra, given a choice of Weyl chamber in $\mathfrak{t}$ of $\mathfrak{k}$, assume that $\{\lambda\}$ is the set of fundamental weights in $\mathfrak{t}^*$, $\{\alpha\}$ is the set positive roots, and W is the Weyl group. Let k denote the rank of $\mathfrak{k}$.

For a character of $\mathfrak{t}$, l, define the symbolic notation

$$z^l = \exp(2\pi i l(x)), \quad z_1^l = \exp(2\pi i l(y)) \tag{6.1}$$

where x, y are $\mathfrak{t}$-valued forms of even order, e.g., $x = x' + x''$ with $x' \in \mathfrak{t}$ and x'' a $\mathfrak{t}$-valued curvature two form. Denote the number of simple roots by m, and the set of positive roots by $\Delta^+(K)$, the set of fundamental weights by $\{\lambda\}$.

Proposition 6.1 (Fundamental Formula)**.**

$$\begin{aligned}
&\frac{1}{z^\rho z_1^\rho \prod_{\alpha\in\Delta^+(K)}(1-z^{-\alpha})(1-z_1^{-\alpha})} \sum_{w,v\in W} \frac{(-1)^{m+\sigma(w)+\sigma(v)}}{\prod_\lambda (1-z^{-w\lambda}z_1^{-v\lambda})}\\
&= \sum_{w,v\in W} \frac{1}{\prod_\alpha(1-z^{-w\alpha})\prod_\lambda(1-z^{w\lambda}z_1^{v\lambda})\prod_\alpha(1-z_1^{-v\alpha})}\\
&= 0.
\end{aligned} \tag{6.2}$$

Pf: Step 1): Apply the well known formula that

$$\sum_{w(\alpha)<0} w(\alpha) = w(\rho) - \rho$$

where ρ is the half sum of the positive roots, to obtain

$$\begin{aligned}
&\frac{1}{\prod_\alpha(1-z^{-w\alpha})\prod_\lambda(1-z^{w\lambda}z_1^{v\lambda})\prod_\alpha(1-z_1^{-v\alpha})}\\
&= \frac{(-1)^{\sigma(w)+\sigma(v)}}{z^{(-w(\rho)+\rho)}z_1^{(-v(\rho)+\rho)}\prod_\alpha(1-z^{-\alpha})\prod_\lambda(1-z^{w\lambda}z_1^{v\lambda})\prod_\alpha(1-z_1^{-\alpha})}.
\end{aligned} \tag{6.3}$$

Next observe that $\sum_\lambda \lambda = \rho$, or $\sum w(\lambda) = w(\rho)$. Likewise for v, thus

$$\prod_\lambda (1-z^{w\lambda}z_1^{v\lambda}) = (-1)^m z^{w\rho}z_1^{v\rho}\prod_\lambda(1-z^{-w\lambda}z_1^{-v\lambda}). \tag{6.4}$$

Denote the left side of the equation (6.2) by $L.H.$, after substituting the last expression into (6.3) to get

(6.5)

$$L.H. = \frac{1}{z^\rho z_1^\rho \prod_\alpha(1-z^{-\alpha})\prod_\alpha(1-z_1^{-\alpha})} \sum_{w,v\in W} (-1)^{m+\sigma(w)+\sigma(v)} \frac{1}{\prod_\lambda(1-z^{-w\lambda}z_1^{-v\lambda})}.$$

First we rename the index v to vw, this is legitimate for obvious reason, then

$$
\begin{aligned}
&\sum_{w,v\in W}(-1)^{\sigma(w)+\sigma(v)}\frac{1}{\prod_\lambda(1-z^{w\lambda}z_1^{v\lambda})}\\
&=\sum_{w,v\in W}(-1)^{\sigma(w)+\sigma(vw)}\frac{1}{\prod_\lambda(1-z^{w\lambda}z_1^{vw\lambda})}\\
&=\sum_{w,v\in W}(-1)^{\sigma(v)}\frac{1}{\prod_\lambda(1-z^{w\lambda}z_1^{vw\lambda})}.
\end{aligned}\tag{6.6}
$$

Step 2): Next we shall verify that the last sum vanishes.

It is well known that $\{\lambda\}$ spans the positive Weyl cone $\mathfrak{t}^+$, and the set of cones of the form $w(\mathfrak{t}^+), \forall w \in W$ spans $\mathfrak{t}$, and there is no overlapping in the interiors of those cones.

Let $\mathcal{S}$ denote the set of all possible subsets of $\{\lambda\}$, and $\forall S \in \mathcal{S}$, let $|S| = \#S$, it satisfies $1 \leq |S| \leq m$. The maximum number is m.

From [M], one obtains the following relation:

$$
\sum_{S\in\mathcal{S},w\in W}(-1)^{|S|}\frac{1}{\prod_{\lambda\in S}\Big(1-\exp< w\lambda,2\pi ix>\Big)}=1.\tag{6.7}
$$

From this we deduce that

$$
\begin{aligned}
&(-1)^m\sum_{w\in W}\frac{1}{\prod_\lambda\Big(1-\exp< w\lambda,2\pi ix>\Big)}\\
&=1-\sum_{|S|<k,w\in W}(-1)^{|S|}\frac{1}{\prod_{\lambda\in S}\Big(1-\exp< w\lambda,2\pi ix>\Big)}.
\end{aligned}\tag{6.8}
$$

Let $x = t + v(s)$ with $z = e^{2\pi it}, z_1 = e^{2\pi is}$, we have

$$
\exp< w\lambda, 2\pi ix >= z^{w\lambda}z_1^{vw\lambda}.
$$

Summing over $v \in W$ with sign $(-1)^{\sigma(v)}$, we have

$$
\sum_{w,v\in W}\frac{(-1)^{k+\sigma(v)}}{\prod_\lambda(1-z^{w\lambda}z_1^{vw\lambda})}=\sum_{v\in W}(-1)^{\sigma(v)}-\sum_{w,v\in W,|S|<k}\frac{(-1)^{|S|+\sigma(v)}}{\prod_{\lambda\in S}(1-z^{w\lambda}z_1^{vw\lambda})},\tag{6.9}
$$

the first term in the last line is 0. When $|S| < k$, those weights in S span a face of the Weyl chamber $\mathfrak{t}^+$. The face, or S, is fixed by a non-trivial subgroup of W. The subgroup is denoted by W_S which is the Weyl group of a subgroup of K. And the subgroup $\mathrm{Ad}_w W_S$ fixes the set of weights $w(S)$.

Let W be divided into cosets by $\operatorname{Ad}_w W_S$ and let $[v_0]$ denote the coset, then for a fixed S with $|S|<k$, one has

$$
\begin{aligned}
&\sum_{w,v\in W}\frac{(-1)^{\sigma(v)}}{\prod_{\lambda\in S}\left(1-z^{w\lambda}z_1^{vw\lambda}\right)}\\
&=\sum_{[v_0]\in W/\operatorname{Ad}_w W_S, w\in W}\ \sum_{v\in\operatorname{Ad}_w W_S}\frac{(-1)^{\sigma(v)}}{\prod_{\lambda\in S}\left(1-z^{w\lambda}z_1^{vw\lambda}\right)}\\
&=\sum_{[v_0]\in W/\operatorname{Ad}_w W_S, w\in W}\ \sum_{v\in\operatorname{Ad}_w W_S}\frac{(-1)^{\sigma(v)}}{\prod_{\lambda\in S}\left(1-z^{w\lambda}z_1^{v_0w\lambda}\right)}\\
&=0
\end{aligned}
\tag{6.10}
$$

because $\sum_{v\in\operatorname{Ad}_w W_S}(-1)^{\sigma(v)}=0$. Thus we have the desired claim. QED

7. Affine Weyl subgroups and the dual Coxeter numbers

In order to apply the formula proved in the last section, we need to know more about those fixed points of type 2) and 3) occurring along an affine wall of C.

Some familiarity with the affine Kac-Moody algebra is required here, see [K] for reference.

Let (p, I, z) be a point which defines a fixed point in X_N. Assume that $\mu(p)/k = \tilde{\phi}(z) \in (\partial C, 1)$, i.e. that fixed point is of type 2) and 3). Let μ denote $\mu(p)$.

Conventions:

In the following, we treat the case when μ is of level 1 to simplify the notation. The general case follows after replacing μ by μ/k.

If $\mu \in \partial C$, there are two possibilities:

a). $\mu \in C \setminus C^{\text{aff}}$, i.e., μ is not on the wall defined by $\theta = 1$,

b). $\theta(\mu) = 1$, or $\mu \in C^{\text{aff}}$.

In the first case, W_μ is generated by reflections of a subset of simple roots, and $(LG)_\mu$ is generated by all the $\mathfrak{g}_\beta$ such that β is a simple root vanishing at μ. In particular, $(LG)_\mu$ is a subgroup of G and W_μ is a subgroup of the regular Weyl group W.

In case of b), the affine root $\alpha_0 = \delta - \theta$ plays an important role. And the reflection with respect to $\alpha_0 = 0$ or $\theta = 1$ is the composition of the reflection defined by $\theta = 0$ and a translation element.

To understand this and the group $(LG)_\mu$ more thoroughly, we need a few things from the theory of affine Lie algebra.

7.1. A few facts on affine Lie algebra.

Proposition 7.1. *Let Δ^+ be the set of simple roots of $\mathfrak{g}$ with respect to the cone spanned by the alcove C, $\{\Lambda_i\}$ be the set of fundamental weights of $\mathfrak{g}$.*

a). If μ is not on the affine wall, then

$$\Delta^+_\mu = \{\beta | \beta(\mu) = 0, \beta \in \Delta^+\}$$

is a set of simple roots for $(l\mathfrak{g})_\mu$. The fundamental weights of $(l\mathfrak{g})_\mu$ is the orthogonal projection with respect to the form $(\cdot|\cdot)$ of $\{\Lambda_\beta\}$ to the linear span of Δ^+_μ.

b). If μ is on the affine wall defined by $\theta = 1$, let

$$\Delta^0_\mu = \{\beta | \beta(\mu) = 0, \beta \in \Delta^+\},$$

$$\Delta^+_\mu = \{\delta - \theta\} \cup \Delta^0_\mu$$

is the set of simple roots. The fundamental weights are given by the orthogonal projection of

$$-\mu, \quad \Lambda_\beta - a_i^{\vee} \mu, \, \beta \in \Delta^+_\mu$$

to the linear span of Δ^+_μ.

In both cases, the group $(LG)_\mu$ is connected and its Weyl group is the subgroup in W^{aff} generated by reflections using elements in Δ^+_μ.

Remark: The fundamental weights of $(l\mathfrak{g})_\mu$ may not be weights of $\mathfrak{g}$. Nevertheless, they are in the rational span of Δ^+.

Pf: For a), the Lie algebra $(l\mathfrak{g})_\mu$ is a subalgebra of $\mathfrak{g}$, generated by $\mathfrak{g}_\beta$, $\beta \in \Delta^+_\mu$. The assertions are well known facts about finite dimensional semi-simple algebras.

For part b), first we verify the assertion about the fundamental weights.

$$
\begin{aligned}
&2(\Lambda_\beta - a^{\mathrm{v}}_\beta\mu|\alpha)/(\alpha|\alpha) = 2(\Lambda_\beta|\alpha)/(\alpha|\alpha) = \delta_{\beta\alpha}, \quad \alpha \in \Delta^0_\mu;\\
&(-\mu|\alpha) = 0, \quad \alpha \in \Delta^0_\mu;\\
&(-\mu| - \theta) = 1;\\
&(\Lambda_\beta - a^{\mathrm{v}}_\beta\mu|\theta) = a^{\mathrm{v}}_\beta - a^{\mathrm{v}}_\beta = 0,
\end{aligned} \tag{7.1}
$$

the last equality is due to the fact that $(\Lambda_\beta|\theta) = a^{\mathrm{v}}_\beta$. Thus, the given set is dual to Δ^+_μ, and therefore its orthogonal projection is the set of fundamental weights of $(l\mathfrak{g})_\mu$.

The best way to see Δ^+_μ of part b) is a set of simple roots is to use the theory of affine Lie algebras, from which we learn that

$$\Delta_\mu = \{\alpha_0 := \delta - \theta\} \cup \Delta^+$$

form a set of simple roots for the algebra $\mathfrak{g}^{\mathrm{aff}}$. Now it is known that any subset of simple roots is a set of simple roots for the subalgebra generated by the subset. Therefore,

$$\Delta^+_\mu = \{\alpha_0\} \cup \Delta^0_\mu$$

is a set of simple roots. On the other hand,

$$(\delta|\delta) = 0, \quad (\delta|\alpha) = 0, \ \forall\alpha \in \Delta,$$

see [K,Ch.6]. Thus the inner product of a pair in Δ' is the same as that of the corresponding pair in Δ^+_μ. In particular, the Dynkin diagrams formed by Δ' and Δ^+_μ are the same. Therefore, Δ_μ form a set of simple roots for the subalgebra $(l\mathfrak{g})_\mu$.

From the characterization of the simple roots of $(LG)_\mu$, it is clear that its Weyl group is generated by reflections using Δ^+_μ. In case μ is on an affine wall, the reflections are with respect to $\{\alpha = 0|\alpha \in \Delta^0_\mu\}$ and $\theta = 1$, as desired.

The connectedness is based on the well known argument in Lie theory. Decompose $(LG)_\mu = \cup K_i$ into connected component, and K_0 contains I. For $g \in K_i$, $\mathrm{Ad}_g K_0 = K_0$. Multiplying g with an element in K_0 if necessary, we can assume that $\mathrm{Ad}_g T = T$ where T is the maximal torus. Therefore, g is in the Weyl group of K_0, therefore $g \in K_0$. QED

7.2. **The half sum of positive roots of $(LG)_\mu$.** Let Δ_μ be as before, and $\{\lambda_i\}$ be the set of fundamental weights of $(l\mathfrak{g})_\mu$. We have seen that λ_i is given by the orthogonal projection of $\Lambda_\beta - a^{\mathrm{v}}_\beta\mu$.

Let ρ_μ be the half sum of positive roots of $(l\mathfrak{g})_\mu$. For finite dimensional Lie algebra, it is well known that

$$\rho_\mu = \sum_i \lambda_i. \tag{7.2}$$

The following is as important as the fundamental formula:

Proposition 7.2. *Let ρ be the half sum of positive roots of $\mathfrak{g}$, $\rho_{\mathrm{aff}} = \rho + h^{\mathrm{v}}\Lambda_0$, where h^{v} is the dual Coxeter number defined by $h^{\mathrm{v}} - 1 = \sum_{i=1,\ldots,l} a^{\mathrm{v}}_i$, then the following holds:*

1). If $<\theta,\phi>< 1$:

$$w(\rho_\mu) - \rho_\mu = w(\rho) - \rho \mod \mathbb{Z}\delta, \quad \forall w \in W_\mu$$

and

$$e^{2\pi i(<vk\phi-\rho+v\rho-v\rho_\mu,t>)}=e^{2\pi i<k\phi+\rho_\mu,t>}. \tag{7.3}$$

2). If $<\theta,\phi>=1$ is on the affine wall of C,

$$r(\rho_\mu)-\rho_\mu=r(\rho)-\rho \quad \mod \mathbb{Z}\delta$$

where r is a reflection defined by a simple root of $\mathfrak{g}$. Let r_θ be the reflection with respect to the affine wall $\phi(\theta^\vee)=1$,

$$r_\theta(\rho_\mu)-\rho_\mu=r_\theta(\rho)-\rho+h^\vee\theta,$$

where $\nu:\mathfrak{g}^{\text{aff}}\to\mathfrak{g}^{\text{aff}*}$ is the map induced by the bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}^{\text{aff}}$.

And when t is restricted to the lattice $\frac{M^*}{k+h^\vee}$, with M^* being the dual of the long root lattice,

$$e^{2\pi i(<vk\phi+\rho-v\rho+v\rho_\mu,t>)}=e^{2\pi i<k\phi+\rho_\mu,t>},\forall v\in W_\mu^0$$

where W_μ^0 is the subgroup of W generated by the reflections with respect to $\alpha\in\Delta_\mu^0$ and θ. It is isomorphic to W_μ.

Pf: 1). It is enough to verify that for reflection r_α, $\alpha\in\Delta_\mu^+$. In this case, α is also a simple root of $\mathfrak{g}$. Thus,

$$r(\rho_\mu)-\rho_\mu=-\alpha=r(\rho)-\rho.$$

In this case, all the simple roots in Δ_μ^+ vanish at $\tilde\phi$ by definition of $(l\mathfrak{g})_\mu$. Therefore, $v\phi=\phi$, the eq. (7.3) holds as a consequence of the identity just proved.

2). If r is generated by a simple root of $\mathfrak{g}$, which is the case if $\alpha\in\Delta_\mu^0$, then the previous argument works. If $r=r_\theta$, $-\theta$ is a simple root of $(l\mathfrak{g})_\mu$ but not of $\mathfrak{g}$, we obtain

$$r_\theta(\rho_\mu)-\rho_\mu=\theta \qquad \mod \mathbb{Z}\delta.$$

On the other hand

$$r_\theta(\rho)-\rho=-<\rho,\theta^\vee>\theta=-(h^\vee-1)\theta,$$

where the last equation is from the definition of $h^\vee$:

$$h^\vee=1+\sum_i a_i^\vee=1+<\rho,\theta^\vee>.$$

Now

$$r_\theta k\phi=k\phi-k<\phi,\theta^\vee>\theta=k\phi-k\theta,$$

thus we obtain

$$\begin{aligned}&e^{2\pi i<r_\theta k\phi-\rho+r_\theta\rho-r_\theta\rho_\mu,t>}\\ &\quad=e^{2\pi i<k\phi-(k+h^\vee)\theta-\rho_\mu,t>}\end{aligned} \tag{7.4}$$

which equals $e^{2\pi i<k\phi-\rho_\mu,t>}$ on the lattice since

$$<(k+h^\vee)\theta,t>\in\mathbb{Z},\quad \forall t\in\frac{M^*}{k+h^\vee}.$$

The above holds now for the generators in W_μ^0, therefore it holds on the lattice $\frac{M^*}{k+h^\vee}$ for all W_μ^0.

The isomorphism between W_μ and W_μ^0, the only difference among the generators is the form has a reflection w.r.t. $\theta=1$ while the latter has one w.r.t. $\theta=0$. QED

8. Further orbifold complications

In order to apply fixed point principle to the space X_N, we need better understanding of the action by T on the normal bundles to the fixed point set of all three types.

8.1. **Weights on the toric variety $X_{\mathfrak{g}}$.** We divide the discussion on the normal bundles according to the types of the fixed point sets.

We continue to use the convention from the last section.

8.2. **When μ is on $(\partial C, 1)$.** This is the more interesting case. Recall that $X_{\mathfrak{g}}$ is constructed as a global orbifold toric variety. First we investigate what is the stabilizer and the weights at a point on $X_{\mathfrak{g}}$.

From the last section, we have learned that the fundamental weights of $(LG)_\mu$ is given by orthogonal projections of the following vectors:

$$\Lambda_\beta - a^{\mathrm{v}}_\beta \mu, \quad \beta \in \Delta_\mu$$

together with μ if $\alpha_0 = \delta - \theta \in \Delta_\mu$. In the above, Λ_β is a fundamental weight of $\mathfrak{g}$ as well since β is a simple root of $\mathfrak{g}$. Because $X_{\mathfrak{g}}$ is an orbifold, the polytope C is not a simple convex polytope (see [O] for definition) with respect to the weight lattice M generated by $\{\Lambda_\beta\}$, but rather it is a simple polytope in the larger lattice M' generated by $\{\Lambda_\beta / a^{\mathrm{v}}_\beta\}$. The larger lattice defines a unique covering of T, T' so that $\pi T' \to T$ has the quotient M'/M as its kernel.

The dual lattice of M is given by the coroot lattice

$$N = \sum \mathbb{Z}\alpha^{\mathrm{v}}_i.$$

The dual lattice of M' is given by the sublattice

$$N' = \sum \mathbb{Z}(a^{\mathrm{v}}_i \alpha^{\mathrm{v}}_i).$$

So $T = \mathbb{R}^l / N$ and $T' = \mathbb{R}^l / N'$.

The polytope C is integral with respect to M', and it is actually a simple simplex. Thus it defines smooth toric variety X' with respect to the group $(T')^{\mathbb{C}}$, X' is in fact the projective space $\mathbb{C}P^l$, and $X_{\mathfrak{g}} = X'/\ker\pi$, see [Od, p96].

Proposition 8.1. *On $X' \simeq \mathbb{C}P^l$, assume $z' \in \phi^{-1}(\partial C)$. The stabilizer of z', T'_z and the weights of the action by T'_z on the normal bundle is given by the following:*

1). If $\mu = \phi(z')$ is not on the affine wall, the stabilizer is given by

$$\sum_{\beta\in\Delta_\mu} \mathbb{R}\beta^{\mathrm{v}} / (\sum_{\beta\in\Delta_\mu} \mathbb{Z} a^{\mathrm{v}}_\beta \beta^{\mathrm{v}}).$$

The weights are given by the orthogonal projection to $\nu(\mathfrak{t}'_z)$ of

$$\Lambda_\beta / a^{\mathrm{v}}_\beta, \forall \beta \in \Delta_\mu.$$

2). If μ is on the affine wall, recall $\Delta_\mu = \{-\theta\} \cup \Delta^0_\mu$ and $\Delta^0_\mu = \{\beta \in \Delta_+ | \beta(\mu) = 0\}$. The coroot $\theta^{\mathrm{v}} \in N'$ by definition is $\sum_i a^{\mathrm{v}}_i \alpha^{\mathrm{v}}_i$.

In particular $\sum_\beta \mathbb{Z} a^{\mathrm{v}}_\beta \beta^{\mathrm{v}}$ is a sublattice of N', where $a^{\mathrm{v}}_\theta = 1$. The stabilizer again is given by

$$\sum_{\beta\in\Delta_\mu} \mathbb{R}\beta^{\mathrm{v}} / (\sum_{\beta\in\Delta_\mu} \mathbb{Z} a^{\mathrm{v}}_\beta \beta^{\mathrm{v}}) \subset T'.$$

The weights of the stabilizer are the orthogonal projection to $\nu(\mathfrak{t}'_z)$ of

$$\{-\mu\} \cup \{\Lambda_\beta / a^{\mathrm{v}}_\beta - \mu \big| \beta \in \Delta^0_\mu\}.$$

Pf: The simplex C defines a polarization and a moment map ϕ which has C as its image. Let ∂C_μ be the smallest face passing μ and V_μ be the smallest linear space containing $\partial C_\mu - \mu$. If $\phi(z') = \mu \in \partial C$, the image of $d\phi(z')$ is exactly V_μ, and the subspace perpendicular to $V_\mu \subset \mathfrak{t}^*$ is the image under ν of the stabilizer $\mathfrak{t}'_z$.

On the other hand, the face ∂C_μ is defined by $\cap_{\beta\in\Delta_\mu}\beta^{-1}(0)$, if μ is not on the affine wall. And given by

$$\theta^{-1}(1) \cap_{\beta\in\Delta^0_\mu} \beta^{-1}(0),$$

if μ is on the affine wall. In either cases,

$$V_\mu^\perp = \nu(\sum_{\beta\in\Delta_\mu} \mathbb{R}\beta^{\mathrm{v}}).$$

Therefore, the Lie algebra of the stabilizer $\mathfrak{t}'_z$ is of the desired form. Clearly the lattice $\sum_{\beta\in\Delta_\mu}\mathbb{Z}a^{\mathrm{v}}_\beta\beta^{\mathrm{v}}$ is in $\mathfrak{t}'_z$, and in fact it is $\mathfrak{t}'_z\cap N'$. Thus the stabilizer of z' in T'_z is $\mathfrak{t}'_z/(\mathfrak{t}'_z\cap N')$ whose explicit form is given by the proposition.

To understand the claim on the weights of the action by the stabilizer on the normal bundle, we recall first that each point on the toric variety X', the neighborhood is constructed as follows:

Let A be the tangent cone of the simplex C at the point $\phi(z')$, A is a convex cone. Take the semi-group $\sigma_\mu = N'\cap A$. Because C is a simple simplex, it is easy to see that

$$\sigma_\mu = \sum_i \mathbb{Z}_{\geq 0}\eta_i + \sum_j \mathbb{Z}\xi_j,$$

where $\{\eta_i\}\cup\{\xi_j\}$ is a base of the lattice N'. Since $\sum_i \mathbb{Z}\xi_i$ is a sublattice, it is given by the lattice points in the maximal linear subspace contained in A, V_μ. Then the action of T' on a neighborhood of z' in the toric variety X' is given by

$$t(z_1,...,z_m,w_1,...,w_n) = (t^{\eta_1}z_1,...,t^{\eta_m}z_m,t^{\xi_n}w_1,...,t^{\xi_n}w_n),$$

the point z' correspond to a point with $w_j = 0, z_i \neq 0$ which also defines the fixed point set of the subgroup T'_z. The above facts about toric varieties can be found in Ch. 1.2 and Ch. 2.4 in [Od]. From this it is easy to read off the weights by the action of T'_z near $w_i = 0$. They are given by the restriction of those weights ξ_i to T'_z. Or the projections to $\nu^{-1}\mathfrak{t}'_z$ of $\{\xi_j\}$.

What are those weights $\{\eta_i\}\cup\{\xi_j\}$? First select a lattice point $\Lambda_i/a^{\mathrm{v}}_i$ on the affine subspace spanned by the smallest face ∂C_μ passing μ, then

$$\{\eta_k\} = \{\Lambda_k/a^{\mathrm{v}}_k - \Lambda_i/a^{\mathrm{v}}_i \in V_\mu\}, \quad \{\xi_j\} = \{\Lambda_j/a^{\mathrm{v}}_j - \Lambda_i/a^{\mathrm{v}}_i \notin V_\mu\}.$$

On the other hand, if one replace $\Lambda_i/a^{\mathrm{v}}_i$ by a point on ∂C_μ, such as μ itself, the projection to the orthogonal complement of V_μ does not change. Therefore, the weights of the action by T'_z can also be given by the orthogonal projections of

$$\{\Lambda_j/a^{\mathrm{v}}_j - \mu \big| \Lambda_j/a^{\mathrm{v}}_j - \mu \notin V_\mu\}.$$

In the above construction, $\{\Lambda_j/a^{\mathrm{v}}_j = 0$ is allowed to account for the weight which is the projection of $-\mu$. QED

Definition 8.1. *Let ϵ^{v} be an element of the coroot lattice so that it is given by $(1/n)\sum_{\alpha_i\notin\Delta^0_\mu} a^{\mathrm{v}}_i\alpha^{\mathrm{v}}_i$, where $n \geq 1$ and no fraction of ϵ is in the coroot lattice.*

Obviously, for $su(l+1)$, there is only one choice $n = 1$.

Notice the definition of n depends on Δ^0_μ.

Corollary 8.1. *1). When $<\phi,\theta><1$ where $\phi=\phi(z)\in\mathfrak{t}^*$, the group T'_z/T_z is given by the finite group*

$$\exp(\sum_{\alpha_i\in\Delta_\mu} b_i\alpha_i^{\text{v}}),\quad 0\le b_i<a_i^{\text{v}}.$$

2). When $<\phi,\theta>=1$ where $\phi=\phi(z)$, the subgroup T'_z/T_z is given by

$$\exp(b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0} b_i\alpha_i^{\text{v}}),\quad 0\le b<n,\quad 0\le b_i<a_i^{\text{v}}.$$

Pf: The group in question satisfies $T'_z/T_z\simeq N/N'$. From the expression of N,N', we can identify the elements in the kernel of the map $T'_z\to T_z$ easily.

The second assertion is based on the simple observation that

$$n\epsilon^{\text{v}}=\theta^{\text{v}}\qquad\text{mod}\sum_{\beta\in\Delta_\mu^0}\mathbb{Z}a_\beta^{\text{v}}\beta^{\text{v}}.$$

Now we claim $\{n\epsilon^{\text{v}}\}\cup\{a_\beta^{\text{v}}\beta^{\text{v}}\}_{\beta\in\Delta_\mu^0}$ spans N' as well. To verify the claim, let m be in the sub-lattice $N\cap(\sum_{\alpha_i\in\Delta_\mu^0}\mathbb{R}\alpha_i^{\text{v}}+\mathbb{R}\theta^v)$, then $m=\sum_{\alpha_i\in\Delta_\mu^0}r_i\alpha_i^{\text{v}}+r\theta^{\text{v}}$, let Λ_i act on both sides, by assumption $\Lambda_i(m)\in\mathbb{Z}$, therefore $m_i=r_i+ra_i^{\text{v}}=\Lambda_i(m)\in\mathbb{Z}$. So $m=\sum_{\alpha_i\in\Delta_\mu^0}(m_i-ra_i^{\text{v}})\alpha_i^{\text{v}}+r\theta^{\text{v}}$. Replace m by $m'=-\sum_{\alpha_i\in\Delta_\mu^0}ra_i^{\text{v}}\alpha_i^{\text{v}}+r\theta^{\text{v}}=r\sum_{\alpha_j\notin\Delta_\mu^0}a_j^{\text{v}}\alpha_j^{\text{v}}$, where in the last equation the definition of θ^{v} is used. Clearly m' is in the same sub-lattice as m, and m' is a integer multiple of ϵ^{v} because $ra_j^{\text{v}}\in\mathbb{Z}$. This proves the claim.

From there, it is easy to identify what $N/N'\simeq T'_z/T_z$ is. QED.

8.3. The transformation of T'_z/T_z.

Lemma 8.1. *Under the action by W_μ, the isotropy group T'_z/T_z transforms into itself.*

For ϕ with $<\phi,\theta>=1$ and $\mu=(\phi,1)$,

$$e^{2\pi i<\phi,w(b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}})>}=e^{2\pi i<\phi,wb\epsilon^{\text{v}}>}=e^{2\pi i<\phi,b\epsilon^{\text{v}}>}.$$

Pf: The first one is easy to verify using reflections defined by simple roots of $(LG)_\mu$, to be more explicit:

$$r_\beta(b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}})=b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}}-<b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}},\beta>\beta^{\text{v}}$$

where β is either θ or in Δ_μ^0. In either case, because the coefficient of β^{v} above is in $\mathbb{Z}$, after mod out the lattice defining T'_z, it is clear that the element above is in T'_z/T_z.

For the second part, first observe that $e^{2\pi i<\phi,\beta^{\text{v}}>}=1$ if $\beta\in\Delta_\mu^0$ or $\beta=\theta$. So the reflections does not change the value, hence it is invariant under W_μ. Or

$$e^{2\pi i<\phi,w(b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}})>}=e^{2\pi i<\phi,b\epsilon^{\text{v}}+\sum_{\alpha_i\in\Delta_\mu^0}b_i\alpha_i^{\text{v}}>}=e^{2\pi i<\phi,b\epsilon^{\text{v}}>}.\quad\text{QED}$$

8.4. **The relations among the groups T_z,T'_z and the maximal torus in $(LG)_\mu^{\text{ss}}$.** We have just studied the relation between T_z,T'_z. A third Abelian group is the maximal torus S of $(LG)_\mu^{\text{ss}}$, the three share the same Lie algebra. The difference is the defining lattice.

Lemma 8.2. *The three groups are related as:*

$$T_z' \to S \to T_z$$

where each arrow is a covering. When $\theta(\phi) \neq 1$, $S \simeq T_z$. *When* $\theta(\phi) = 1$, $S/T_z \simeq \mathbb{Z}/n\mathbb{Z}$ *where* n *is defined before.*

Pf: If $\theta(\phi) \neq 1$, the coroot lattice of $(LG)_\mu^{\mathrm{ss}}$ is given by $\{\alpha^\vee\}_{\alpha\in\Delta_\mu^+}$, where each $\alpha^\vee$ is also a coroot of $\mathfrak{g}$. So the lattice defining T_z is the same as that defines S.

If $< \theta, \phi >= 1$, the coroots are $\{-\theta^\vee\} \cup \{\alpha^\vee\}_{\alpha\in\Delta_\mu^0}$ which form the lattice defining S. On the other hand, the lattice defining T_z is generated by $\epsilon^\vee$ and $\{\alpha^\vee\}_{\alpha\in\Delta_\mu^0}$, therefore the claim is verified. QED

9. A couple of integration formulas

One of the key steps in prove the cancellation formula is the following evaluation of certain differential forms on a space which is a fiber bundle. To be more precise, let $Z \to F \to E$ be an sequence so that $F = Z/S$ and $E = Z/K$ where S is a the maximal torus of K and K admits a local free action on Z.

We shall use the notations introduced in Sect 1 on the connection dPA defining the vertical and horizontal parts of F. Let B, R be the vertical and horizontal part of dPA respectively.

There is one exception here, S is used instead of T_z and K is used instead of $\mathcal{K}''$.

Proposition 9.1. *1). Let ϵ be a weight of S, then the bundle $Z \times_S \mathbb{C}$ with $(ps, v) \simeq (p, s^\epsilon v)$ defines a line bundle on $Z/S = F$ with curvatures given by $< \epsilon, dPA >$. The Chern class is given by $< \epsilon, B + R >$.*

2). The following holds: $\mathrm{Td}(TF) = \mathrm{Td}(T^H F) \cdot \mathrm{Td}(T'' F), \mathrm{Td}(T^H F) = \pi^* \mathrm{Td}(E)$ *and*

$$\mathrm{Td}(T''F) = \prod_{\tau \in \Delta_+} \frac{-i/2\pi < \tau, B + R >}{(1 - e^{i/2\pi < \tau, B+R>})}$$

where Δ_+ is the set of positive roots of $\mathrm{Lie}\mathcal{K}$*.*

3). Localization for a family:

$$\int_{\pi^{-1}([p])} \mathrm{Td}(TF) e^{i/2\pi<\epsilon, B+R>} = \mathrm{Td}(E) \sum_{u \in W(K)} \frac{e^{i/2\pi<u\epsilon, B>}}{\prod_{\tau \in \Delta_+}(1 - e^{-i/2\pi<\tau, uB>})}.$$

(This equation and the next should be viewed as identities about differential forms.)

4).

$$\begin{aligned} &\int_{\pi^{-1}([p])} \frac{\mathrm{Td}(TF)}{\prod_\epsilon (1 - e^{2\pi i<\epsilon, t + 1/4\pi^2 R + 1/4\pi^2 B>)})} \\ &= \mathrm{Td}(E) \sum_{u \in W(K)} \frac{1}{\prod_{\tau \in \Delta_+}(1 - e^{<\tau, i/2\pi uB>})(1 - e^{2\pi i<\epsilon, t + 1/4\pi^2 uB>})}. \end{aligned} \tag{9.1}$$

Pf: The first part repeats Prop. 5.1. Since $TF = T^H F \times T'' F$, the Todd class satisfies the equality

$$\mathrm{Td}(F) = \mathrm{Td}(TF) = \mathrm{Td}(T^H F) \times \mathrm{Td}(T'' F).$$

On the other hand, as discussion in Sect.1, $T''F \simeq Z \times_S \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{s}^\perp \subset \mathfrak{k}$. Hence

$$T''F = \oplus_{\tau \in \Delta_+} Z \times_S \mathbb{C},$$

where $-\tau$ is the character of S acting on $\mathbb{C}$, the sign reflects the choice of the complex structure on $X \times \mathbb{X}$. According to Part 1), the Chern class of the line bundle is represented by $- < \tau, B + R >$ and the Todd class of $T''F$ is the product of the Todd class of the line bundle corresponding to each positive root τ. Thus the expression is verified.

To see 3), we employ the localization of equivariant cohomology class

$$\int_{K/S} \prod_{\tau \in \Delta_+} \frac{-2\pi i < \tau, 1/4\pi^2 R + X >}{(1 - e^{2\pi i<\tau, X + 1/4\pi^2 R>})} e^{2\pi i<\epsilon, X + 1/4\pi^2 R>} = \sum_{u \in W(K)} \frac{e^{2\pi i<\epsilon, uX>}}{\prod(1 - e^{2\pi i<\tau, uX>})}, \tag{9.2}$$

the above is an identity for analytic functions in $X, \forall X \in \mathfrak{s}$. If we plug the $\mathfrak{s}$-valued 2-form B , instead of X, we end up with an equality of forms. This is exactly the claim of Part 3).

To see the claim of Part 4, inserting $0 < r < 1$ and introducing the multi-index $\vec{n} \in \mathbb{Z}^m$ with $m = \#\{\epsilon\}$, and $\vec{\epsilon} = (\epsilon_1, ...\epsilon_m)$. Then

$$
\begin{aligned}
&\int_{\pi^{-1}([p])} \frac{\mathrm{Td}(TF)}{\prod_\epsilon (1 - re^{2\pi i<\epsilon, t+1/4\pi^2 R + 1/4\pi^2 B>})} \\
&= \mathrm{Td}(T^H F) \sum_{\vec{n}} r^{|\vec{n}|} e^{2\pi i \vec{n}\cdot\vec{\epsilon} t} \int_{\pi^{-1}([p])} \mathrm{Td}(T'' F) e^{<\vec{n}\cdot\vec{\epsilon}, i/2\pi(R+B)>} \\
&= \mathrm{Td}(T^H F) \sum_{\vec{n}} r^{|\vec{n}|} e^{\vec{n}\cdot\vec{\epsilon} t} \sum_{u\in W(K)} \frac{e^{<\vec{n}\cdot\vec{\epsilon}, i/2\pi u B>}}{\prod(1 - e^{i/2\pi<\tau, uB>})} \qquad (9.3) \\
&= \mathrm{Td}(T^H F) \sum_{u \in W(K)} \frac{1}{\prod(1 - e^{<\tau, 1/4\pi^2 uB>}) \prod_\epsilon (1 - re^{2\pi i<\epsilon, t+1/4\pi^2 uB>})},
\end{aligned}
$$

once the formula is established for $0 < r < 1$, we can take limit $r \to 1$. QED

10. T, G-spaces and the consequences of the main result

Before we tackle the technical difficulty, the main cancellation, let's first see a couple of consequences of the main theorem.

There are two groups of results presented here, one is in the general case and the other is the holomorphic case.

10.1. **Passing from T-modules to G-modules.** If V_T is a T-module with weight vectors in $\mathfrak{t}_+^*$, then one can apply the holomorphic induction to get a G-module from it. Namely, let

$$V_T = \oplus_{\lambda \in \mathfrak{t}_+^*} m_\lambda \mathbb{C} v_\lambda$$

where v_λ is a weight vector with weight λ and $m_\lambda \in \mathbb{Z}_+$ the multiplicity of the weight λ, then define

$$V_G = \oplus_{\lambda \in \mathfrak{t}_+^*} m_\lambda V_\lambda$$

where V_λ is the unique highest weight G-module with λ as the highest weight.

The above is just the usual holomorphic induction.

Apply this construction to $H^0(X_N, L_N)$ to get a G-module which is denoted by $V_G(X_N, L_N)$.

The long passage we have taken that started from the $\widetilde{LG}$-module $H^0(X, L)$ to $H^0(X_N, L_N)$, and then $V_G(X_N, L_N)$ has some advantage. It is relatively easy to construct X_N which should be viewed as the compactified quotient of X by the maximal Borel $B^+ \subset LG^{\mathbb{C}}$. We could have taken the quotient by $B_I^+ \subset B^+$ which is I at a fixed point on the circle. It would be more difficult to find its compactification, and will be done on another occasion. Although the space $X_N \times_T G$ which will be discussed more in the next section can be thought of as poor man's version of X/B_I^+.

10.2. **The character functions of T and G modules.** The character functions of the modules V_T, V_G are related. Let

$$\chi_G(g) = \mathrm{tr}(t|_{V_G}), \quad \chi_T(t) = \mathrm{tr}(t|_{V_T}).$$

It is known that $\chi_G(g)$ is a class function, so its values are determined by the restriction to $t \in T$. As an easy application of the famed Weyl character formula, one has the following:

$$\chi_G = \sum_{w \in W} w \cdot \frac{\chi_T}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})}. \tag{10.1}$$

10.3. **Relating T, G-spaces.** The description above has a generalization. Suppose that P is a compact symplectic manifold (or orbifold) and V is a complex line bundle on it, and the pair admits an action by T, so that the action by T on P is Hamiltonian. Assume the above data fit together in the sense of geometric quantization, c.f. [GS]. Then one can define the following T-equivariant Riemann-Roch:

$$\mathrm{RR}_P^T(t) = \int_P \mathrm{Td}(TP)\mathrm{Ch}(V)(t), \tag{10.2}$$

the above is also the equivariant index of a *spin*$^{\mathbb{C}}$-complex. Via the fixed point formulas of Atiyah-Bott-Segal-Singer, the above can be written as contribution as $\sum_F \mathrm{FC}_F(t)$ where $\{F\}$ is set of connected components of fixed points, $\mathrm{FC}_F(t)$ is an integral on F involving Td, Ch and equivariant classes of the normal bundle of F in P. The exact expression will be given later.

By assumption on P, there is a moment map $\phi : P \to \mathfrak{t}$. Suppose that $\phi(P) \subset \mathfrak{t}^+$ which is the positive Weyl chamber of $\mathfrak{g}$, we can associate a G-space with P, $P \times_T G$, a G-bundle with V over $P \times_T G$, $\pi^*(V)/T$ where $\pi : P \times G \to P$ is the projection, the action by $t \in T$ on $\pi^*(V)$ is given by the following:

$$(p, g, v) \in \pi^*(V) \mapsto (tp, tg, t^{\phi(p)}v). \tag{10.3}$$

The following proposition is an easy exercise, as an application of the Atiyah-Bott-Segal-Singer fixed point formula on both T and $P \times_T G$:

Proposition 10.1.

$$\mathrm{RR}^G_{P\times_T G}(t) = \sum_{w\in W} \frac{\mathrm{RR}^T_P(wt)}{\prod_{\alpha\in\Delta^+}(1-(wt)^{-\alpha})} \tag{10.4}$$

where the left side represents the G-equivariant Riemann-Roch number associated with $(P \times_T G, \pi^*(V)/T)$.

10.4. **Contribution from fixed point sets.** Given a connected component F of T-fixed point set in $X_{\mathfrak{t}} = \mu^{-1}(\mathfrak{t} \times \{k\})$, define

$$\mathrm{FC}_F(t) = \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L|_F)}{\det(1 - t^{-1}e^{-\Omega})|\mathrm{nor}^{\mathbb{C}}(F, X_{\mathfrak{t}})}, \tag{10.5}$$

where $\mathrm{Td}(F), \mathrm{Ch}(L_F, t)$ are Todd class and T-equivariant Chern class of $TF, L|_F$ respectively; the denominator is the standard equivariant class in the finite dimensional fixed point formula of the complex normal bundle of F in $X_{\mathfrak{t}}$. The existence of the complex structure on the normal bundle has been shown in Section 1.

10.5. **Coefficients of modular transformations.** Use the notations introduced in Section 1, and let χ_a be the character function of the highest weight G-module defined by $a \in P_+$. The following is in [K, Ch. 13]:

Proposition 10.2.

$$\frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_{\lambda\in P^k_+} \big(\chi_b \cdot \chi_{\bar a} \cdot D^2\big)\big(e^{2\pi i\nu^{-1}(\frac{\lambda+\rho}{k+h^\vee})}\big) = \delta_{b,a}, \tag{10.6}$$

where $\bar a$ is the weight whose highest weight module is contragredient to the one defined by a. (or $\bar a = w_L(-a)$ with w_L being the longest element in W).

10.6. **Consequence of the main theorem.** Assume the main theorem, how can we determine the function $\mathrm{RR}(Y)$ from its values on the subset $\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P^k_+\}$? We will prove Cor. 1.1 here.

Lemma 10.1. *1).*

$$\mathrm{RR}^T(X_N, L_N)(t) = \sum_{a\in P^k_+} m_a t^a$$

where P^k_+ is the set of weights in kC.

2). $m_a = \mathrm{RR}(\mathcal{M}_a, L_a)$ the Riemann-Roch number of the pair $(\mathcal{M}_a, L_a)$ where $\mathcal{M}_a = \phi^{-1}(a)/T$ and L_a is the induced orbifold line bundle on $\mathcal{M}_a$.

3). $\mathrm{RR}^G(Y, L_Y) = \sum_{a\in P^k_+} m_a\chi_a$ where χ_a is the character function of the highest weight G-module defined by a.

The first assertion follows from an important fact that $m_a = 0$ if a is outside the image of $k\phi$. We already know that the image lies in kC, so a must be in kC if $m_a \neq 0$. That a has to be a weight, because $\mathrm{RR}^T(X_N, L_N)(t)$ is a function on T.

Part 2 follows the affirmative solution (the Abelian case) to a conjecture by Guillemin-Sternberg.

The last part uses part 1 together with the Weyl character formula:

$$\begin{aligned} \mathrm{RR}^G(Y, L_Y) &= \sum_{w \in W} w \frac{\mathrm{RR}^T(X_N, L_N)}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} \\ &= \sum_{w \in W} w \frac{\sum_{a \in P_+^k} m_a t^a}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} \\ &= \sum_{a \in P_+^k} m_a \chi_a. \quad \text{QED} \end{aligned} \tag{10.7}$$

Pf. of Cor. 1.1: Let $\mathrm{RR}(Y) = \sum_{a \in P_+} m_a \chi_a$, then from the discussion of the T and G-spaces, we know that

$$\mathrm{RR}(X_N)(t) = \sum_{a \in P_+} m_a t^a.$$

Thus only those $\{a\}$ inside kC will occur, since $\phi(X_N) \subset kC$. So one can write $\mathrm{RR}(Y)$ as $\sum_{a \in P_+^k} m_a \chi_a$. Multiplying $D^2 \chi_{\bar{a}}$ on both sides and sum over

$$\tau \in \{e^{2\pi i \nu^{-1} \frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}$$

to get

$$\begin{aligned} m_a &= \frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_\tau (\mathrm{RR}(Y) \cdot D^2 \cdot \chi_{\bar{a}})(\tau) \\ &= \frac{(-1)^l}{\left|\frac{M^*}{(k+h^\vee)M}\right|} \sum_\tau \chi_{\bar{a}}(\tau) \cdot D^2(\tau) \Big(\sum_{\{F | \mu(F) \in kW(C^{\mathrm{int}})\}} \mathrm{FC}_F(\tau) + \mathcal{R}(\tau) \Big). \end{aligned} \tag{10.8}$$

QED

10.7. **Application to the holomorphic case.** Assume X is holomorphic, and it satisfies the conditions that μ is both transversal and proper.

The result in this subsection does not rely on the main theorem of [C1]. First we write down the character formula/fixed point formula of the module $V_G(X_N, L_N)$ in terms of the fixed points on X.

Theorem 10.1. *Assume $H^0(X_N, L_N) = \mathrm{RR}(X_N, L_N)$, which holds if the higher cohomology groups vanish.*

1). $H^0(X_N, L_N) \simeq \sum_{a \in P_+^k} m_a \mathbb{C}_a$ where C_a is the T-module on which T acts with weight a.

2). Let $\chi_G(e^{it}), t \in \mathfrak{t}$ be the character of the G-module $V_G(X_N, L_N)$, then

$$\chi_G(\tau) = (\sum_F \mathrm{FC}_F + \mathcal{R})(\tau), \quad \tau \in \{e^{2\pi i \nu^{-1} \frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\} \tag{10.9}$$

Furthermore, the multiplicity is given by of an irreducible component with the highest weight a is given by the same expression as m_a in Eq. (10.8).

Pf:

$$\begin{aligned}\chi_T(X_N, L_N)(t) &:= \mathrm{tr}(t|H^0(X_N, L_N)) \\ &= \sum_i (-1)^i \mathrm{tr}(t|H^i(X_N, L_N)) \\ &= \mathrm{RR}(X_N, L_N) \\ &= \sum_{a \in P_+^k} m_a t^a.\end{aligned} \tag{10.10}$$

The first part follows.

Using the earlier result relating $\chi_T(X_N, L_N)$ and χ_G, we have

$$\begin{aligned}\chi_G &= \sum_{w \in W} w \frac{\chi_T(X_N, L_N)}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} \\ &= \sum_{w \in W} w \frac{\mathrm{RR}(X_N, L_N)}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})} \\ &= \mathrm{RR}(Y),\end{aligned} \tag{10.11}$$

where the last equality follows from Prop. 10.1. From there, the assertion is shown to be true by the main theorem and Cor. 1.1. QED

10.8. **The character function of the $\widetilde{LG}$-modules.** Now we can derive a character formula for the representations of $\widetilde{LG}$ on $H^0(X, L)$, assuming the main theorem of [C1].

The derivation here requires certain formal manipulations. The formal aspect to the approach here can be traced back to the derivation of Weyl-Kac character formula.

Let $\chi_{\widetilde{LG}}(X, L)(t)$ be the trace of t acting on the part of $H^0(X, L)$ generated by the highest weigh modules of level k. The qualifier here about the trace is included because we do not know at this point whether $H^0(X, L)$ is generated by the highest weight modules. If $H^0(X, L)$ is a representation of finite energy, then automatically it has the desired quality, see [PS].

Theorem 10.2. *Define the regular and reduced Weyl-Kac denominators as follows*

$$\mathbf{D}^{\mathrm{WK}} = \prod_{\tilde\alpha > 0}(1 - e^{-\tilde\alpha}), \quad \mathbf{D}_0^{\mathrm{WK}} = \prod_{\tilde\alpha > 0, \tilde\alpha \notin \Delta^+}(1 - e^{-\tilde\alpha})$$

where $\tilde\alpha$ in the first function runs through all the positive roots of $\widetilde{l\mathfrak{g}}$, while the second does not contain the positive roots $\mathfrak{g}$, Δ^+. Under the same assumptions as in Thm 10.1, the character $\chi_{\widetilde{LG}}(X, L)(t)$ is given by

$$\begin{aligned}&\chi_{\widetilde{LG}}(X, L)(t) \\ &= \sum_{F \in \mathcal{F}} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L|_F, t)}{\det(I - t^{-1}e^{-\Omega})|_{\mathrm{nor}^{\mathbb{C}}(F,X)}} + \sum_{w \in W^{\mathrm{aff}}/W} w \frac{\mathcal{R}(\tau)}{\mathbf{D}_0^{\mathrm{WK}}}.\end{aligned} \tag{10.12}$$

where $\mathrm{nor}^{\mathbb{C}}(F, X)$ *is the complex normal bundle of F in X.*

Remark: It is known that Weyl-Kac formula has a formal flavor to it. The infinite sum and product in the denominator above reflects that.

Pf: It is clear that $\mathbf{D}^{\mathrm{WK}} = D \cdot \mathbf{D}_0^{\mathrm{WK}}$ with D as in Section 1.1.

Given $a \in P_+^k$, there is a unique highest weight $\widetilde{LG}$-module at level k, $\tilde{V}_a$. The character $\mathrm{tr}(t|\tilde{V}_a)$ is provided by the Weyl-Kac formula as

$$\tilde{\chi}_a(t) = \sum_{w \in W^{\mathrm{aff}}} w \frac{e^a}{\mathbf{D}^{\mathrm{WK}}}(t).$$

The resemblance to the characters of the G-modules is obvious.

From the representation theory of $\mathfrak{g}_{\mathrm{aff}}$, we learn that each T-module with weights in P_+^k induces such a $\widetilde{LG}$-module, and these are all the irreducible highest weight module of $\widetilde{LG}$ at level k.

Now the part of $H^0(X, L)$ generated by the highest weigh modules of level k is simply

$$\oplus_{a \in P_+^k} m_a \tilde{V}_a,$$

from our identification of all the highest weight vectors in $H^0(X, L)$. Thus we conclude for $t \in \{e^{2\pi i \nu^{-1} \frac{\lambda+\rho}{k+h^\vee}} | \lambda \in P_+^k\}$:

$$\begin{aligned}
\chi_{\widetilde{LG}}(X, L)(t) &= \sum_{w \in W^{\mathrm{aff}}} w \frac{\mathrm{RR}(X_N, L_N)}{\mathbf{D}^{\mathrm{WK}}}(t) \\
&= \sum_{w \in W^{\mathrm{aff}}/W} w \frac{\mathrm{RR}(Y)}{\mathbf{D}_0^{\mathrm{WK}}}(t) \\
&= \sum_{w \in W^{\mathrm{aff}}/W} w \frac{\sum_F \mathrm{FC}_F + \mathcal{R}}{\mathbf{D}_0^{\mathrm{WK}}}(t).
\end{aligned} \tag{10.13}$$

Next we treat the first term. Since each connected component of fixed point sets in $X_{\mathfrak{t}}$ can be written as wF for some $w \in W_{\mathrm{aff}}$, and $F \in \mathcal{F}^1$, it suffices to show

$$w \frac{\sum_{F \in \mathcal{F}^1} \mathrm{FC}_F}{\mathbf{D}_0^{\mathrm{WK}}} = \int_{wF} \frac{\mathrm{Td}(wF)\mathrm{Ch}(L|_{wF}, t)}{\det(I - t^{-1}e^{-\Omega})|_{\mathrm{nor}^{\mathbb{C}}(F,X)}}. \tag{10.14}$$

The map $w : X \to X$ preserves the complex structure, naturally w induces isomorphism between TF, TwF. Also the symplectic form which is defines $c_1(L|_{wF}), c_1(L|_F)$ up to a constant, and is invariant under LG. Hence the two Chern classes are equal under pull-back. The equivariant Chern classes are related by

$$w\mathrm{Ch}(L|_F, t) = e^{2\pi i \mu(wt) + c_1(L|_F)} = w_* e^{2\pi i \mu(wt) + c_1(L|_{wF})}$$

where w_* pulls back forms. And $\mathrm{Td}(TF) = w_* \mathrm{Td}(TwF)$. The only tricky part is the identification of the classes associated with the normal bundles.

Let D be in the denominator of FC_F, i.e.,

$$D(t) = \det(1 - t^{-1}e^{-\Omega})$$

which can be written in terms of Chern roots $\{x_i\}$, the weights $\{\theta_i\}$ and the roots of $\mathfrak{g}$ as

$$\prod_i (1 - t^{-\theta_i} e^{-x_i}) \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}),$$

the index here is finite, since $\mathrm{nor}(F, X_{\mathfrak{t}})$ is of finite dimension.

The manipulation of $w\mathbf{D}^{\mathrm{WK}}$ is similar to the compact case, and we obtain

$$\begin{aligned}
&\det(1-t^{-1}e^{-\Omega})w(\mathbf{D}_0^{\mathrm{WK}})\\
&= w\prod_i(1-t^{-\theta_i}e^{-x_i})\prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathrm{aff}})}(1-t^{-\alpha})\\
&= \prod_i(1-t^{-w\theta_i}e^{-x_i})\prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathrm{aff}})}(1-t^{-w\alpha})\\
&= \det(1-t^{-1}e^{-\Omega})|_{\mathrm{nor}(F,X)},
\end{aligned} \tag{10.15}$$

where we have used the fact that $\mathrm{nor}_x(F,X)\simeq T_x\mu^{-1}(\mathfrak{t})\oplus l\mathfrak{g}/\mathfrak{t}$. We observe that 1). $w^*(x_i^w)=x_i$ where x_i^* is the Chern root of the corresponding line bundle at wF. 2). Each $(l\mathfrak{g})_\alpha$ in $(l\mathfrak{g}/\mathfrak{t})$ induces a trivial bundle over F by group action, hence it has no curvature. 3). The weights on wF on $\mathrm{nor}(wF,X)$ is given by $w\{\theta_i\}\cup w\Delta^+(\mathfrak{g}_{\mathrm{aff}})$.

After observing the above, immediately we obtain

$$R.H. = w_*\det(1-t^{-1}e^{-\Omega})|\mathrm{nor}(wF,X).$$

Thus we complete the proof. QED

11. The proof of the main cancellation

Proposition 11.1. *Let F_h denote a connected component of fixed point sets (here h may be I). Suppose $\mu = \mu(F_h) \in k(\partial C, 1)$ and is preserved by W_μ^{aff}. Let w be a lifting in W_μ^{aff} of $[w] \in W_\mu / W(\mathcal{K}_h)$, $wF_h = F_{wh}$ (or denoted by F_h^w) is another component of fixed point sets with the same value under μ.*

1). If $\mu(F_h) = k(\phi, 1)$ with $\phi \in \partial C$, but $\phi \notin C^{\mathrm{aff}} = \{\theta = 1\}$, then

$$\sum_{v \in W} v \frac{\sum_{[w] \in W_\mu / W(\mathcal{K}_h)} \mathrm{FC}_{wF_h}}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})}(t) = 0.$$

2). If $\phi \in \partial C \cap \{\theta = 1\}$ and vFC_{wF_h} has no pole on $\{e^{2\pi i \nu^{-1}\frac{\lambda+\rho}{k+h^{\mathrm{v}}}} | \lambda \in P_+^k\}$, then

$$\sum_{v \in W} v \frac{\sum_{[w] \in W_\mu / W(\mathcal{K}_h)} \mathrm{FC}_{wF_h}}{\prod_{\alpha \in \Delta^+}(1 - e^{-2\pi i \alpha})} = 0 \quad \text{on} \quad \frac{M^*}{k + h^{\mathrm{v}}}.$$

Pf:

Step 1: Lifting action by T to a covering group $T_p \times T_Z'$.

A key ingredient in Atiyah-Bott-Segal-Singer's fixed point formula is the contribution from the normal bundles which appear in the denominators of an integral on the fixed point set. Ignoring for a moment the complications from orbifolds, the fixed point formula requires evaluating:

$$\mathrm{FC}_F(g) = \int_F \frac{\mathrm{Td}(TF)\mathrm{Ch}(L|_F, g)}{\det(1 - g^{-1}e^{-\Omega})}, \tag{11.1}$$

where Ω is $i/2\pi$ times the curvature operator of the complex normal bundle.

In what follows, we will evaluate the integral above for wF, wF_h and for $g = t, vt$ where w, v are elements in Weyl subgroups specified later. Furthermore, we need to include the consideration that $\{F\}$ are orbifolds, if F is of type 2, 3, i.e., $\phi(F) \in \partial C$.

Suppose $v, w \in W_\mu$, then the decomposition of $t = (t_{wp}, t_z^w) \in \mathfrak{t}_{wp} \oplus \mathfrak{t}_z$ has the following property as shown in Section 4:

$$(vt)_{wp} = wt_p, \quad (vt)_z^w = vt_z + (vt_p - t_p) \in \mathfrak{t}_z.$$

Use t_z to denote $vt_z + (vt_p - t_p)$, there should be no confusion. And denote by $e^{2\pi i t_z'}, e^{2\pi i s'} \in T_z'$ certain lifting of $e^{2\pi i t_z} \in T_z$, $e^{2\pi i s} \in T_z \cap T_p$ respectively.

1). If μ is on the affine wall, the lifting of $ve^{2\pi i t} \in T$ to $T_{wp} \times T_z'$ are given by

$$(e^{2\pi(wt_p + ws)}, e^{2\pi i(t_z' - s' + \sum b_i \alpha_i^{\mathrm{v}} + b\epsilon^{\mathrm{v}})}) \in T_{wp} \times T_z' \text{ with } e^{2\pi i(ws)} \in wI_p^0 = I_{wp}^0, \tag{11.2}$$

where $0 \leq b_i < a_i^{\mathrm{v}}, 0 \leq b < n$ with n, ϵ the same as in Corollary8.1.

2). If μ is not on the affine wall, then the lifting are given by

$$(e^{2\pi(wt_p + ws)}, e^{2\pi i(t_z' - s' + \sum b_i \alpha_i^{\mathrm{v}})}) \in T_p \times T_z \text{ with } e^{2\pi i(ws)} \in wI_p^0 = I_{wp}^0. \tag{11.3}$$

Lemma 11.1. *Let g be a lifting as above, and $v \in W_\mu^0$ which is the subgroup of W isomorphic to W_μ^{aff}. For $< \phi, \theta >= 1$,*

$$g^\phi = e^{2\pi i < \phi, vt + b\epsilon >} = e^{2\pi i < v\phi, t + b\epsilon >};$$

for $< \phi, \theta > \neq 1$,

$$g^\phi = e^{2\pi i < \phi, vt >}.$$

Pf: First of all, ϕ is a weight on T_p, since p is fixed by T_p, and ϕ is the character of L at p. Now use Lemma 8.2, Part 2 of Corollary 8.1 and the fact that ϕ is a weight on S in the notation there, we have the assertion. QED

Step 2: Defining the denominators.

Let g denote a lifting just described. Recall F_h is a strata associated with F and is fixed by T and h in the isotropy group of F. From now, we use F_h instead of F and treat F as the special case $h = I$. We have treated F and F_h separately so far, this leads to repetitions on occasions. We will point out whenever the difference requires additional attention.

The following are various factors which will appear in the denominator as in Eq.(11.1) along $wF_h = F_h^w$. The expressions are obtained using the weights and curvatures computations done in Sections 3-5.

Remark: The signs in front of the weights below are determined by the following consideration:

a). X_N is the reduced space of the product $X \times \mathbb{X}$, for the map $\mu_X - \mu_{\mathbb{X}}$. The weakly symplectic form is $\omega - \omega_{\mathbb{X}}$. Thus one chooses the original J on X, and $-J$ on $T\mathbb{X}$ to make the form semi-positive definite. Hence a negative sign for the weights $\{\beta\}, \{\lambda\}$ on the the normal subbundle $\{l\mathfrak{g}_\mu/\mathfrak{t} \oplus H_z\}$ from $T\mathbb{X}$

b). The expression in the denominator involves $g^{-1}e^{-\Omega}$, therefore another negative sign.

c). The difference in the signs for the term dA_h in various determinants below was referred to in the Remark after Prop. 5.2.

Now we can write down the expressions for the denominators:

1). If μ is on the affine wall, let $\vec{b} = (b, b_1, ..., b_{\mathbf{I}_\mu})$ with $\mathbf{I}_\mu = \#\Delta_\mu^0$ the number of simple roots of $\mathfrak{g}$ which vanish at μ/k.

(11.4)

$$\mathrm{D}_0^w(vt) = \det(1 - g^{-1}e^{-\Omega})|_{N_{wp}\oplus \mathrm{nor}^1(wZ_h, wZ)} = \prod(1 - e^{-2\pi i\gamma(t_p+s+1/4\pi^2\nabla^2)}),$$

$$\widetilde{\mathrm{D}}^w(vt, \vec{b}) = \det(1 - g^{-1}e^{-\Omega})|_{H_z}$$
$$= (1 - e^{-2\pi i w\mu(t'_z - s' + b\epsilon^\vee + \sum_i b_i\alpha_i^\vee - 1/4\pi^2 dA_h)})$$
$$\times \prod(1 - e^{2\pi i w\frac{\Lambda_i - a_i^\vee\mu}{a_i^\vee}(t'_z - s' + b\epsilon^\vee + \sum_i b_i\alpha_i^\vee - 1/4\pi^2 dA_h)}),$$

$$\mathrm{D}^w(vt, b) = (1 - e^{-2\pi i w\mu(t_z - s + b\epsilon^\vee - 1/4\pi^2 dA_h)})$$
$$\times \prod(1 - e^{2\pi i w(\Lambda_i - a_i^\vee\mu)(t_z - s + b\epsilon^\vee - 1/4\pi^2 dA_h)}),$$

$$\mathrm{D}_b^w(vt) = \det(1 - g^{-1}e^{-\Omega})|_{(l\mathfrak{g})_\mu/\mathrm{Lie}\mathcal{K}_{wh}} = \prod_{\beta\in\Delta^+\setminus\Delta^+(\mathcal{K}_{wh})}(1 - e^{2\pi i w\beta(t_p+s+1/4\pi^2 dA_h)}),$$

$$\mathrm{D}_a^w(vt) = \prod_{\alpha\in\Delta^+}(1 - e^{-2\pi i\alpha(vt)}).$$

2). If μ is off the affine wall, let $\vec{b} = (b_1, ..., b_{\mathbf{I}_\mu})$, the one difference is:

$$\widetilde{\mathrm{D}}^w(vt, \vec{b}) = \det(1 - g^{-1}e^{-\Omega})|_{H_z}$$
$$= \prod(1 - e^{2\pi i w\frac{\Lambda_i - a_i^\vee\mu}{a_i^\vee}(t'_z - s' + \sum_i b_i\alpha_i^\vee - 1/4\pi^2 dA_h)}), \tag{11.5}$$
$$\mathrm{D}^w(vt) = \prod(1 - e^{2\pi i w(\Lambda_i - a_i^\vee\mu)(t_z - s + b\epsilon^\vee - 1/4\pi^2 dA_h)}).$$

Obviously in the above definition, when $h = I$, $\mathrm{nor}(wZ, wZ_h)$ is trivial since $Z = Z_h$.

Step 3: Expressing $\mathrm{FC}_{F_h^w}(vt)$.

Continue to let $g \in T_p \times T_z'$ where the first component of g passes $w(hT_p^0)$ in $w(T_p)$, as a lifting of $vt \in T$. This form of lifting appeared in Eq. (11.2), (11.3). In the notations just introduced, we have along the normal bundle of F_h^w

$$\det(1 - g^{-1}e^{-\Omega}) = \mathrm{D}_0^w(g)\mathrm{D}_b^w(g)\widetilde{\mathrm{D}}^w(g) \tag{11.6}$$

by definitions of the factors on the right and the structure of the normal bundle.

The line bundles $L|_{F_h}, L|_{F_h^w}$ (orbifold bundles actually) are related through the map $w : F_h \to wF_h = F_{wh} = F_h^w$ as

$$w^*(L|_{F_h^w}) = L|_{F_h}, \quad w(\phi) = \phi.$$

This relation holds for the pair Z_{wh}, Z_h obviously, and it holds on the quotient T_z because $w \in W_\mu$ normalizes T_z. In terms of F_h, following Lemma 11.1, one obtains

1). when μ is on the affine wall and $w \in W_\mu^{\mathrm{aff}}$,

$$\mathrm{Ch}(L|_{F_h^w}, g) = g^{w\phi}e^{\omega} = g^{\phi}e^{\omega} = e^{2\pi i < kv\phi, t + b\epsilon^{\mathrm{v}} >}e^{\omega}, \tag{11.7}$$

where ω is the symplectic form on X_N restricted to F_h.

2). when μ is off the affine wall,

$$\mathrm{Ch}(L|_{F_h^w}, g) = g^{w\phi}e^{\omega} = g^{\phi}e^{\omega} = e^{2\pi i < kv\phi, t >}e^{\omega}. \tag{11.8}$$

Depending on whether μ is on or off the affine wall, one obtains now an expression of $\mathrm{FC}_{F_h^w}$ in terms of an integral on F_h as follows:

$$\begin{aligned} &\phi(\theta^{\mathrm{v}}) = 1 : \\ &\mathrm{FC}_{F_h^w}(vt) = \frac{1}{|I_p^0||T_z'/T_z|}\sum_g \int_{F_h} \frac{\mathrm{Td}(TF_h)e^{2\pi i < kv\phi, t + b\epsilon^{\mathrm{v}} >}e^{\omega}}{\mathrm{D}_0^w(g)\mathrm{D}_b^w(g)\widetilde{\mathrm{D}}^w(g)}; \\ &\phi(\theta^{\mathrm{v}}) \neq 1 : \\ &\mathrm{FC}_{F_h^w}(vt) = \frac{1}{|I_p^0||T_z'/T_z|}\sum_g \int_{F_h} \frac{\mathrm{Td}(TF_h)e^{2\pi i < kv\phi, t >}e^{\omega}}{\mathrm{D}_0^w(g)\mathrm{D}_b^w(g)\widetilde{\mathrm{D}}^w(g)}. \end{aligned} \tag{11.9}$$

In the above the summation is over all the possible lifting of vt in $w(hT_p^0) \times T_z'$ which is the isotropy group of the strata F_{wh}.

Step 4: Summation over T_z'/T_z.

For an upcoming calculation, it is crucial to replace the fractional weights

$$(\Lambda_i - a_i^{\mathrm{v}}\mu)/a_i^{\mathrm{v}}, \quad \alpha_i \in \Delta_\mu^0$$

by the integral weights on T_z,

$$\Lambda_i - a_i^{\mathrm{v}}\mu, \quad \alpha_i \in \Delta_\mu^0.$$

This amounts to replacing $\widetilde{\mathrm{D}}^w$ by D^w and is an important step. The basic observation is for μ on the wall,

$$|T_z'/T_z| = n \prod_{\alpha_i \in \Delta_\mu^0} a_i^{\mathrm{v}};$$

for μ off the wall,

$$|T_z'/T_z| = \prod_{\alpha_i \in \Delta_\mu^0} a_i^{\mathrm{v}};$$

and more importantly:

$$\sum_{b_i} \frac{1}{\widetilde{\mathrm{D}}^w(g)} = (\prod_{\alpha_i \in \Delta^0_\mu} a_i^{\mathrm{v}}) \frac{1}{\mathrm{D}^w(g)}. \tag{11.10}$$

The last equation is based on two observations:

1). Using geometric series expansion, and the fact that $\sum_{b_i} e^{2\pi i < a, \sum_i b_i \alpha_i^{\mathrm{v}}>}$ is either 0 or $\prod_{\alpha_i \in \Delta^0_\mu} a_i^{\mathrm{v}}$, for all the possible weight a, depending on a is a weight on T'_z but not on T_z, or a is a weight on T_z.

2). Another observation used in the above is

(11.11)

$$e^{-2\pi i < w\mu, t'_z - s' + b\epsilon^{\mathrm{v}} + \sum_i b_i \alpha_i^{\mathrm{v}} - 1/4\pi^2 dA_h>} = e^{-2\pi i < w\mu, t_z - s + b\epsilon^{\mathrm{v}} - 1/4\pi^2 dA_h>},$$
$$e^{2\pi i < w(\Lambda_i - a_i^{\mathrm{v}}\mu), t'_z - s' + b\epsilon^{\mathrm{v}} + \sum_i b_i \alpha_i^{\mathrm{v}} - 1/4\pi^2 dA_h>} = e^{2\pi i < w(\Lambda_i - a_i^{\mathrm{v}}\mu), t_z - s + b\epsilon^{\mathrm{v}} - 1/4\pi^2 dA_h>},$$

due to $\phi(\alpha_i^{\mathrm{v}}) = 0; \Lambda_i(\epsilon^{\mathrm{v}}), \Lambda_i(\alpha_i^{\mathrm{v}}) \in \mathbb{Z}$; also $-\mu, \Lambda_i - a_i^{\mathrm{v}}\mu$ are fundamental weights of T_z, their values at $e^{2\pi i(t'_z - s')} \in T'_z$ are the same at the projections $e^{2\pi i(t_z - s)} \in T_z$.

Thus one has:

$$\begin{aligned}
&\phi(\theta^{\mathrm{v}}) = 1 : \\
&\mathrm{FC}_{F_h^w}(vt) = \frac{1}{n|I_p^0|} \int_{F_h} \frac{w^*\mathrm{Td}(TF_{wh}) e^{2\pi i < kv\phi, t + \epsilon^{\mathrm{v}}>} e^{\omega}}{\mathrm{D}_0^w(vt)\mathrm{D}_b^w(vt)\mathrm{D}^w(vt, b)}; \\
&\phi(\theta^{\mathrm{v}}) \neq 1 : \\
&\mathrm{FC}_{F_h^w}(vt) = \frac{1}{|I_p^0|} \sum_g \int_{F_h} \frac{\mathrm{Td}(TF_h) e^{2\pi i < kv\phi, t>} e^{\omega}}{\mathrm{D}_0^w(vt)\mathrm{D}_b^w(vt)\mathrm{D}^w(vt)}.
\end{aligned} \tag{11.12}$$

Step 5: An identity of Todd classes. Given $w \in W_\mu$, there is the map

$$w : F_h = Z_h/T_z \to wZ_h/T_z = Z_{wh}/T_z.$$

Lemma 11.2.

$$\begin{aligned}
&w^*(\mathrm{Td}(T^H F_{wh})) = \mathrm{Td}(T^H F_h), \\
&w^*(\mathrm{Td}(T'' F_{wh})) = \prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})} \frac{-i/2\pi < w\tau', dA_h >}{1 - e^{i/2\pi < \tau', dA_h>}}.
\end{aligned} \tag{11.13}$$

Remark: The sign convention reflects again the the complex structure chosen on $l\mathfrak{g}/\mathfrak{t}$.

Pf: If $\pi : F_h \to E_h$ as in Section 3, then $\mathrm{Td}(T^H F_{wh}) = \pi^*\mathrm{Td}(E)$. The same holds for F_{wh}, E_{wh}. It is easy to see that $w : Z_h \to Z_{wh}$ is an equivariant isomorphism with respect to the action by $K_h, \mathrm{Ad}_w \mathcal{K}_h = \mathcal{K}_{wh}$ on Z_h, Z_{wh} respectively. Therefore, $\mathrm{Td}(E_h) = w^*\mathrm{Td}(E_{wh})$ whose pull-back to F_h, F_{wh} yield the first equation.

On $T''Z_{wh}$, after decomposing it into a sum of line bundles according to the roots $\{\tau'\}$, the curvature is given by

$$-\oplus < \tau', dA_{wh} >= -\oplus < w\tau', dA_h > .$$

The second identity follows that. QED

Step 6: A sufficient condition for the validity of the cancellation.

First of all, recall

$$w^*\mathrm{Td}(TF_{wh}) = \mathrm{Td}(T^H F_h) w^*\mathrm{Td}(T'' F_{wh})$$

where $\mathrm{Td}(T^H F_h) = \pi^* \mathrm{Td}(E_h)$, with

$$\pi : F_h = Z_h/T_z \to E_h = Z_h/\mathcal{K}'_h.$$

Also it was shown in Prop. 4.4 $D_0^w(vt) = D_0(t)$. We also know both forms below

$$w^* \mathrm{Td}(T^H F_{wh}) = \mathrm{Td}(T^H F_h), \quad e^\omega$$

are the pull-back of forms on E_h, because they are null in the fiber direction of the map π. Also clearly that $D_a^w(vt)$ is constant with respect to the integration variable. Therefore, in order to evaluate

$$\sum_g \int_{F_h} \frac{w^* \mathrm{Td}(TF_{wh}) e^{2\pi i < kv\phi, t + b\epsilon^{\mathrm{v}} >} e^\omega}{\mathrm{D}_0^w(vt) \mathrm{D}_b^w(vt) \mathrm{D}^w(vt, b)}$$

(for $< \phi, \theta >= 1$) or

$$\sum_g \int_{F_h} \frac{w^* \mathrm{Td}(TF_{wh}) e^{2\pi i < kv\phi, t >} e^\omega}{\mathrm{D}_0^w(vt) \mathrm{D}_b^w(vt) \mathrm{D}^w(vt)}$$

(if $< \phi, \theta > \neq 1$), one can pull $D_0^w(vt), \mathrm{Td}(T^H F_h)$ and e^ω out, when integrating along $\pi^{-1}([p])$; those factors except $D_a^w(vt)$ are independent of $v \in W_\mu^0, w \in W_\mu$. Thus to prove the proposition, which involving evaluating a sum over W_μ of the above integrals, it suffices to prove

(11.14)

$\phi(\theta^{\mathrm{v}}) = 1:$

$$\sum_{w \in W_\mu/W(\mathcal{K}_h); v \in W_\mu^0} \int_{\pi^{-1}([p])} \frac{w^* \mathrm{Td}''(TF_{wh}) e^{2\pi i k \phi(t_p + s + b\epsilon^{\mathrm{v}})}}{\mathrm{D}_a^w(vt) \mathrm{D}_b^w(vt) \mathrm{D}^w(vt, b)} = 0, \quad \text{on} \quad \frac{M^*}{k + h^{\mathrm{v}}};$$

$\phi(\theta^{\mathrm{v}}) \neq 1:$

$$\sum_{w \in W_\mu/W(\mathcal{K}_h); v \in W_\mu^0} \int_{\pi^{-1}([p])} \frac{w^* \mathrm{Td}(T'' F_{wh}) e^{2\pi i k \phi(t_p + s + b\epsilon^{\mathrm{v}})}}{\mathrm{D}_a^w(vt) \mathrm{D}_b^w(vt) \mathrm{D}^w(vt)} = 0.$$

In the above, a lifting of each $w \in W_\mu/W(\mathcal{K}_h)$ to W_μ is fixed and denoted by the same.

Step 7: Turn the integrals along the fiber to a sum via equivariant cohomology.

Recall from Section 3 $dA_h = B_h + R_h$, where B_h is the horizontal part of the curvature, R_h is the vertical part tangent to $\pi^{-1}([p])$. Recall also that $\pi^{-1}([p])$ is a finite quotient of $\mathcal{K}'_h/T_z$. Hence the integrals can be pulled to $\mathcal{K}'_h/T_z$.

We will continue to use the same notations for the pull-back to $\mathcal{K}'_h/T_z$ of various curvature forms on $\pi^{-1}([p])$. In Eq.(11.14), the term $\mathrm{D}_a^w(vt)$ is a constant on $\mathcal{K}'_h/T_z$. The integral of the rest were calculated using equivariant cohomology. As a straightforward application of Formula 4 in Prop. 9.1, the answers are

$\phi(\theta^{\mathrm{v}}) = 1:$

$$(11.15) \quad \begin{aligned} &\int_{\mathcal{K}'_h/T_z} \frac{w^* \mathrm{Td}(T'' F_{wh}) e^{2\pi i k \phi(t_p + s + b\epsilon^{\mathrm{v}})}}{\mathrm{D}_a^w(vt) \mathrm{D}_b^w(vt) \mathrm{D}^w(vt, b)} \\ &= \frac{e^{2\pi i k \phi(t_p + s + b\epsilon^{\mathrm{v}})}}{\mathrm{D}_a^w(vt)} \sum_{u \in W(\mathcal{K}_h)} \frac{1}{\prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})} (1 - e^{2\pi i < w\tau', 1/4\pi^2 uB >}) \mathrm{d}_b^w \mathrm{d}^w}, \end{aligned}$$

where

(11.16)
$$\mathrm{d}_b^w = \prod_\beta (1 - e^{2\pi i w\beta(t_p+s+1/4\pi^2 uB_h)}),$$
$$\mathrm{d}^w = (1 - e^{-2\pi i w\mu(t_z-s+b\epsilon^\vee+1/4\pi^2 uB_h)}) \prod (1 - e^{2\pi i w(\Lambda_i - a_i^\vee \mu)(t_z-s+b\epsilon^\vee-1/4\pi^2 uB_h}),$$

in the above the $\{\beta\}$ are in $\Delta_\mu^+ \setminus \Delta^+(\mathcal{K}_{wh})$. (In applying Prop. 9.1 above, we treat factors $\mathrm{D}_b^w(vt)\mathrm{D}^w(vt,b)$ collectively as $\prod_\epsilon (1 - e^{2\pi i<\epsilon, it+1/4\pi^2 R_h+1/4\pi^2 B_h>})$; and use $w\tau'$ instead of τ there.)

For the other case $\mu(\theta^\vee) \neq 1$, the only difference is in the definition of d_b^w,

$$\mathrm{d}^w = \prod_{i \in \mathbf{I}_\mu} (1 - e^{2\pi i(w\Lambda_i)(t_z-s+b\epsilon^\vee-1/4\pi^2 uB_h}). \tag{11.17}$$

Step 8: The first lucky break.

We will now simplify the expression just obtained to be in a position to apply the fundamental formula of Section 6.

The first break comes when we group the factor involving the positive roots $\{\tau'\} = \Delta^+(\mathcal{K}_{wh})$, with the one involving $\{\beta\} = \Delta_\mu^+ \setminus \Delta^+(\mathcal{K}_{wh})$, we realize that the first factor can be made look just like the second one. To be more specific:

$$\begin{aligned} &\prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})} (1 - e^{i/2\pi<w\tau', uB_h>}) \\ &= \prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})} (1 - e^{2\pi i<w\tau', t_p+s+1/4\pi^2 uB_h>}). \end{aligned} \tag{11.18}$$

There are two reasons for the above equation: 1). $e^{2\pi i u(t_p+s)} = e^{2\pi i(t_p+s)} \in hT_p^0$, since $u \in W(\mathcal{K}_h)$ and $\mathcal{K}_h$ commutes with hT_p^0 by its definition; 2). $e^{2\pi i w\tau'(t_p+s)} = 1$, the adjoint action by hT_p^0 on Lie$\mathcal{K}_h$ is I since the two commute by the definition of Lie$\mathcal{K}_h$, and $w\tau'$ is a root of $\mathcal{K}_h$.

In this form, the product is clearly in the same species as d_b^w.

Step 9: The second break and the finale. The denominator needs to be written in a form so we can apply Prop.6.1. The denominator acquires an extra factor after integration as shown in the previous two steps. So the total is

$$\mathcal{D}_v^w = \mathrm{D}_a^w(vt)\mathrm{D}_b^w(vt)\mathrm{D}^w(vt,b) \prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})} (1 - e^{2\pi i<w\tau', t_p+s+1/4\pi^2 uB_h>})$$

if $\phi(\theta^\vee) = 1$. For $\phi(\theta^\vee) \neq 1$, $\mathcal{D}_v^w$ is defined the same way except $\mathrm{D}^w(vt,b)$ is replaced by $\mathrm{D}^w(vt)$ as defined in Step 2 above.

According to the previous step, $\mathrm{D}_b^w(vt)$ which has β running over $\Delta_\mu^+ \setminus \Delta^+(\mathcal{K}_{wh})$ can be written together with $\prod_{\tau' \in \Delta^+(\mathcal{K}_{wh})}(1 - e^{2\pi i<w\tau', t_p+s+1/4\pi^2 uB_h>})$ as

$$\prod_{\beta \in \Delta_\mu^+} (1 - e^{2\pi i<w\beta, t_p+s+1/4\pi^2 uB_h>}), \tag{11.19}$$

which can be further written as

$$\prod_{\beta \in \Delta_\mu^+} (1 - e^{2\pi i<w\beta, t_p+s+b\epsilon^\vee+1/4\pi^2 uB_h>}), \tag{11.20}$$

for β is a weight of T_z, therefore is trivial on the T_z'/T_z in which $e^{2\pi i b\epsilon^\vee}$ lies.

Now the full expression of $\mathcal{D}_v^w$ is given by

$$\mathcal{D}_v^w = \prod_{\beta\in\Delta_\mu^+}(1 - e^{2\pi i<w\beta,t_p+s+b\epsilon^\vee+1/4\pi^2uB_h>}) \prod_{\alpha\in\Delta^+}(1 - e^{-2\pi i<v\alpha,t>}) \times \prod_\lambda (1 - e^{2\pi i(<v\lambda,t>-<w\lambda,t_p+s+b\epsilon^\vee+1/4\pi^2uB_h>)}) \tag{11.21}$$

where λ runs through the fundamental weights of $(l\mathfrak{g})_\mu$. If $\phi(\theta^\vee) \neq 1$, just remove the term $b\epsilon^\vee$ in the above. Also

$$\begin{aligned} e^{2\pi i<w\beta,t_p+s+b\epsilon^\vee>} &= e^{2\pi i<w\beta,u(t_p+s+b\epsilon^\vee)>}, \\ e^{-2\pi i<w\lambda,t_p+s+b\epsilon^\vee>} &= e^{-2\pi i<w\lambda,u(t_p+s+b\epsilon^\vee)>}, \end{aligned} \tag{11.22}$$

for $u \in W(\mathcal{K}_h)$, this property can be deduced from Lemma 8.1 and the fact that $u|_{hT_p^0} = I$, for $u \in W(\mathcal{K}_h)$ commutes with hT_p^0 as $\mathcal{K}_h$ does.

As mentioned earlier, w is a lifting of $W_\mu/W(\mathcal{K}_h)$ in W_μ, and $u \in W(\mathcal{K}_h)$. The product wu on $\mathfrak{g}$ runs through the whole W_μ. Or uw on $\mathfrak{g}^*$ goes through W_μ. Denote the combined by $w \in W_\mu$.

Let $y = t_p + s + b\epsilon^\vee + 1/4\pi^2B_h$. Finally, for $v \in W_\mu^0, w \in W_\mu$, and m being the rank of Lie$\mathcal{K}_h$ we have the following after pulling out some of the exponential terms:

$$\begin{aligned} \mathcal{D}_v^w &= \prod_{\beta\in\Delta_\mu^+}(1 - e^{2\pi i<w\beta,y>}) \prod_{\alpha\in\Delta^+}(1 - e^{-2\pi i<v\alpha,t>}) \\ &\quad \times \prod_\lambda (1 - e^{2\pi i(<v\lambda,t>-<w\lambda,y>)}) \\ &= (-1)^{m+\sigma(w)+\sigma(v)} e^{2\pi i(<w\rho_\mu-\rho_\mu-w\sum\lambda,y>+<\rho-v\rho+v\sum\lambda,t>} \\ &\quad \times \prod_{\beta\in\Delta_\mu^+}(1 - e^{-2\pi i<\beta,y>}) \prod_{\alpha\in\Delta^+}(1 - e^{-2\pi i<\alpha,t>}) \prod_\lambda (1 - e^{-2\pi i(<v\lambda,t>-<w\lambda,y>)}), \end{aligned} \tag{11.23}$$

in the exponent of the first term above, there is the cancellation

$$\rho_\mu - \sum \lambda = 0.$$

The final calculation is

$$\begin{aligned} &\sum_{v\in W_\mu^0, w\in W_\mu} \frac{e^{2\pi i<vk\phi,t+b\epsilon^\vee>}}{\mathcal{D}_v^w} \\ &= \frac{e^{2\pi i<-\rho_\mu,y>}}{\prod_{\alpha\in\Delta^+}(1 - e^{2\pi i<v\alpha,t>})\prod_{\beta\in\Delta_\mu^+}(1 - e^{-2\pi i<\beta,y>})} \\ &\quad \times \sum_{v\in W_\mu^0, w\in W_\mu} \frac{(-1)^{m+\sigma(w)+\sigma(v)} e^{2\pi i(<vk\phi,t+b\epsilon^\vee>-<\rho-v\rho+v\rho_\mu,t>)}}{\prod_\lambda (1 - e^{-2\pi i(<v\lambda,t>-<w\lambda,y>)})}, \end{aligned} \tag{11.24}$$

after observing $e^{2\pi i<vk\phi,b\epsilon^\vee>} = e^{2\pi i<k\phi,b\epsilon^\vee>}$, hence it can be taken outside the summation. It now suffices to show the vanishing of

$$\sum_{v\in W_\mu^0, w\in W_\mu} \frac{(-1)^{m+\sigma(w)+\sigma(v)} e^{2\pi i(<vk\phi+\rho-v\rho+v\rho_\mu,t>)}}{\prod_\lambda (1 - e^{-2\pi i(<v\lambda,t>-<w\lambda,y>)})}. \tag{11.25}$$

The numerator, when $\phi(\theta^{\mathrm{v}}) = 1$ and on the lattice $\frac{M^*}{k+h^{\mathrm{v}}}$ as shown in Prop. 7.2 agrees with

$$e^{2\pi i < k\phi+\rho_\mu, t>}$$

which is independent of v, w and can be pulled outside the summation. When $\phi(\theta^{\mathrm{v}}) \neq 1$, the numerator equals

$$e^{2\pi i < k\phi+\rho_\mu, t>}$$

everywhere. The lattice is invariant under W. Therefore, over this lattice, the vanishing of the above sum is implied by

$$\sum_{v \in W_\mu^0, w \in W_\mu} \frac{(-1)^{\sigma(w)+\sigma(v)}}{\prod_\lambda (1 - e^{-2\pi i (< v\lambda, t> - < w\lambda, y>)})} = 0, \tag{11.26}$$

whose validity is shown by Prop. 6.1. Thus we have completed the proof of Part 1, 2 of Prop. 11.1.

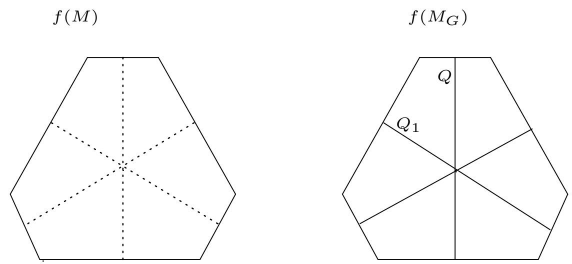

FIGURE 12.1. The extra cuts on $f(M_G)$

12. TWINS AND A NEW SURGERY FORMULA

12.1. A consequences from the proof: Twin pairs of compact G-manifolds. Here we give an easy application for finite dimensional symplectic G-manifolds, or more generally symplectic G-orbifolds.

Let M be a symplectic manifold (or orbifold) with a Hamiltonian G-action, f be the moment map. Suppose that M is compact and $f^{-1}(\mathfrak{t})$ is smooth, i.e. the image of f is transversal to $\mathfrak{t}$, then we can construct a T-orbifold M_N, just as we have done for the LG-space X. Specifically, M_N is constructed as follows: Let $k \in \mathbb{Z}_+$ so that $f(M) \cap \mathfrak{t}_+ \subset kC$. Let $X_{\mathfrak{g}}$ be the same toric variety as in Section 2, with Φ as its moment map. Then

$$M_N = \{(p,q) \in f^{-1}(\mathfrak{t}_+) \times X_{\mathfrak{g}} | f(p) = k\Phi(q)\}/T.$$

Another way to see it is to start with $f^{-1}(\mathfrak{t}_+)$ which is a manifold with non-smooth boundary $f^{-1}(\partial\mathfrak{t}_+)$. Each face Q on $\partial\mathfrak{t}_+$ has a semi-simple stabilizer $G_Q \subset G$ under the adjoint action. The normal subspace to Q is the Lie algebra of a maximal torus of G_Q, T_Q. Then

$$M_N = \cup_Q f^{-1}(Q^{\text{int}})/T_Q.$$

The construction looks similar to symplectic cuts, but the two have quite different properties when one consider surgery formulas for equivariant Riemann-Roch.

The map $\phi(u) = f(p)$ with $u = [p,q]$, is well defined, since f is invariant under T. The variety M_N enjoys the same property as X_N in Section 2.3. The T-fixed points on M_N comes from either fixed points on M, or as a result of surgery along

$$\big(w, \phi^{-1}(\partial\mathfrak{t}_+)\big), \quad w \in W.$$

The latter were studied in Section 3.

Similar to X_N, M_N has degeneracy as a symplectic orbifold and has a T-invariant almost complex structure.

The space

$$M_G = G \times_T M_N$$

is a new G-space. In general, M_G is different from M, when $f(M) \cap \mathfrak{t}$ intersects the boundary of the Weyl chambers. Though the restriction of the image of the moment maps of the two spaces coincide.

The figure 12.1 illustrates the cuts and the intersections of the images of the two spaces with $\mathfrak{t}$.

As already mentioned, the new fixed points all have their images on the boundary of the Weyl chambers. As shown in the proof in the last section, the sum of

contributions of fixed points with the same image vanishes. Therefore, we have the following interesting consequence for the pair:

Corollary 12.1. *The two G-spaces M and M_G have the same G-equivariant Riemann-Roch numbers.*

12.2. **Another consequence: A new surgery formula.** Let M be a symplectic G-orbifold satisfying the transversality condition, i.e. $f(M)$ is transversal to $\mathfrak{t} \subset \mathfrak{g}$, and V be a smooth symplectic subvariety of M. Suppose the T-action preserves V, with $V \subset f^{-1}(\mathfrak{t})$.

Definition 12.1. *1). $(\tilde{U}, G_U)$ is an orbifold chart of U if G_U is an finite group acting with no non-trivial kernel on $\tilde{U}$ so that*

$$\pi : \tilde{U} \to U = \tilde{U}/G_U, \quad \pi(\tilde{p}) = G_U(\tilde{p}) \in U.$$

2). $I_V \subset T$ is the isotropy group of V if for an open set U with $U \cap V \neq \emptyset$, and $I_V \subset G_U$ is the stabilizer of $\pi^{-1}(V \cap U)$. For $\tau \in T$ which fixes V, let τ_V denotes all the local liftings of τ.

From [C], we know that up to isomorphism, the group I_V is independent of the chart, and the open set U. So it is well defined over V.

Remark: The local liftings may not be extended over the entire V, since there could be global monodromy. On the other hand, the following characteristic class over V can be defined, as an average over τI_V:

$$\frac{1}{|I_V|} \sum_{t \in \tau I_V} \int_V \frac{\mathrm{Td}(V)\mathrm{Ch}(L_V)}{\det_{\mathrm{nor}(V,M)}(1 - t^{-1}e^{-\Omega})}$$

where Ω is the $i/2\pi$-curvature operator of the normal bundle of V in M. We want to find another expression for it.

The expression is in terms of the subvarieties in M_G and its lower strata.

For each face Q in $\{w\mathfrak{t}_+ \big| w \in W\}$, Q can be written as wQ' with Q' as a face of $\mathfrak{t}_+$. Thus each Q has a stabilizer in W, denoted by W_Q. Let

$$\begin{aligned} M_Q &= \{(p,q) \in f^{-1}(Q) \times X_{\mathfrak{g}} | f(p) = kw\Phi(q)\}/T; \\ V_Q &= \{(p,q) \in V \times X_{\mathfrak{g}} | f(p) \in Q, \quad f(p) = kw\Phi(q)\}/T. \end{aligned} \tag{12.1}$$

Both M_Q and V_Q are orbifolds, as a consequence of the transversality of $f(M)$ to $\mathfrak{t}$, and both have T-invariant almost complex structures. Therefore one can define Todd class and makes sense of the equivariant normal bundles of V_Q in M_Q.

The readers are warned that there is no natural almost complex structure on the normal bundle of V in $M_{\mathfrak{t}}$.

Lemma 12.1. *1). Let $M_Q^G = G \times M_Q$. Then M_Q, M_Q^G and V_Q are subvarieties in $M_G = G \times_T M_N$.*

2). If F is a subvariety of V_Q, then the isotropy groups, $I(F, V_Q), I(F, M_G)$ of F in V_Q, M_G respectively satisfies the following exact short sequence:

$$1 \to I(V_Q, M_G) \to I(F, M_G) \to I(F, V_Q) \to 1$$

where the second term is the isotropy group of V_Q in M_G. In particular:

$$|I(F, M_G)| = |I(V_Q, M_G)| \cdot |I(F, V_Q)|.$$

Pf: Part 1 is obvious. To see Part 2), let $(\tilde{U}, G_U)$ be an orbifold chart meeting F. If $g \in I(F, M_G)$, is I on $\pi^{-1}(M_G \cap U)$, it remains so on the smaller set $\pi^{-1}(V_Q \cap U)$. Hence $g \in I(F, M_G)$. The next map in the sequence is defined by restriction. If the restriction of g is trivial, it is I on V_Q hence $g \in I(V_Q, M_G)$.

The relation for the size of the group now is an immediate consequence. QED.

Suppose that Q is a sub-face of Q_1, then the above construction shows that V_Q is a subvariety of V_{Q_1}. Obviously

$$V_Q = V_{Q_1} \cap M_Q.$$

Let $\{\Delta\}$ be all the top dimensional faces, i.e. they are $\{w\mathfrak{t}_+\}$, if $Q \subset \Delta$, the normal bundle of V_Q in V_Δ admits a T-action since T acts on both.

The following is a new surgery formula and will be used in the next section to find the remainder term $\mathcal{R}(\tau)$ in the fixed point formula:

Proposition 12.1. *Suppose $\tau \in T$ fixes V. Let $\Lambda^{\max}\mathrm{nor}(V_Q, V_\Delta)$ be the determinant line bundle of $\mathrm{nor}(V_Q, V_\Delta)$, then*

$$\begin{aligned}
&\frac{1}{|I_V|}\sum_{s\in\tau I_V}\int_V \frac{\mathrm{Td}(V)\mathrm{Ch}(L_V)(t)}{\det_{\mathrm{nor}(V,M)}(1-s^{-1}e^{-\Omega})} \\
&= \sum_{\Delta}\sum_{Q\subset\Delta}\frac{1}{|W_Q||I_{V_Q}|}\sum_{s\in\tau I_{V_Q}}\int_{V_Q}\frac{\mathrm{Td}(V_Q)\mathrm{Ch}(L_{V_Q}\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta)(t))}{\det_{\mathrm{nor}(V_Q,M_Q^G)}(1-s^{-1}e^{-\Omega})}
\end{aligned} \tag{12.2}$$

where W_Q, I_{V_Q}, I_V are stabilizer of Q in W and isotropy groups of V_Q, V in M_G, M respectively.

Pf: We prove the statement by first representing both sides in terms of the T-fixed points of V, V_Q. Since T acts on V and its normal bundle, we can apply the localization of equivariant cohomology classes on V to get

$$\begin{aligned}
&\frac{1}{|I_V|}\int_V \frac{\mathrm{Td}(V)\mathrm{Ch}(L_V)}{\det_{\mathrm{nor}(V,M)}(1-t^{-1}e^{-\Omega})} \\
&= \sum_{F\subset V}\frac{1}{|I_V||I(F,V)|}\sum_{t\in\tau I(F,M_G)}\int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{\det_{\mathrm{nor}(F,V)\oplus\mathrm{nor}(V,M)}(1-t^{-1}e^{-\Omega})|_F}
\end{aligned} \tag{12.3}$$

where the denominator can be combined as $\det_{\mathrm{nor}(F,M)}(1-t^{-1}e^{-\Omega})$ obviously.

As for the right hand side, for the same reason as above one can express it as

$$\sum_{\Delta}\sum_{Q\subset\Delta}\sum_{F\subset V_Q}\frac{1}{m}\sum_{t\in\tau I(F,M_G)}\int_F\frac{\mathrm{Td}(F)\mathrm{Ch}(L_F\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta)|_F)}{\det_{\mathrm{nor}(F,M_Q^G)}(1-t^{-1}e^{-\Omega})}$$

where $m = |W_Q||I_{V_Q}||I(F, V_Q)|$.

There are two kinds of fixed point sets on $\{V_Q\}$: those already on V, and those as a result of the cuts along the boundary of Weyl chambers. The first kind have their isotropy groups come with the property that $I(F, M_G) = I(F, M)$ since the cut does not pass through F. The second kind have images under ϕ on the boundary of wC for some w.

Also we may apply Lemma 12.1 2) to the orders of the isotropy groups, so that $|I_{V_Q}||I(F, V_Q)| = |I(F, M_G)|$ independent of V_Q, if $\phi(F)$ is on the boundary of the Weyl chambers.

Next we describe the weights on the various bundles. Without loss of generality, assume Q is a face of $\mathfrak{t}_+$. Let $\mathfrak{g}_\phi$ be the stabilizer of $\phi(F) \in Q \subset \mathfrak{g}$. We claim

that the total contribution of all the terms involving F vanishes, if $\phi(F)$ is on the boundary of the Weyl chambers.

Section 3 analyzed the formation of fixed point sets, here we continue to use the notations.

Let $\mathfrak{g}_\phi^{\mathrm{ss}}$ be the semi-simple part of $\mathfrak{g}_\phi$, $\mathfrak{t}_\phi = \mathfrak{g}_\phi^{\mathrm{ss}} \cap \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_\phi^{\mathrm{ss}}$. Let Λ_ϕ be the set of fundamental weights of $\mathfrak{g}_\phi^{\mathrm{ss}}$, and

$$\Lambda_Q = \{\tilde{\lambda} \in \Lambda_\phi | \tilde{\lambda} \in Q\}, \quad \Lambda_Q^\perp = \Lambda_\phi \setminus \Lambda_Q. \tag{12.4}$$

Here $\mathfrak{t}_\phi$ is identical to $\mathfrak{t}_z$ in Section 3 and Section 4, which is true since both subalgbras are the orthogonal complement of the smallest face containing $\phi(F)$.

As shown in Prop. 4.3, Λ_ϕ induces the following weights along F,

$$s^\lambda = (s_\phi)^{\tilde{\lambda}}$$

for each lifting of s to $(s_p, s_\phi) \in T_p \times T_\phi$. Or in terms of Lie algebra notation:

$$\exp 2\pi i < \lambda, t >= \exp 2\pi i < \tilde{\lambda}, t_\phi >= \exp 2\pi i < \tilde{\lambda}, t - t_p > .$$

The set $\{\lambda | \tilde{\lambda} \in \Lambda_Q^\perp\}$ are all the weights on $\mathrm{nor}(V_Q, V_\Delta)|_F$.

We remark that for each fixed point set F with image on the boundary of $\mathfrak{t}_+^*$, there is a submanifold $Z \subset M$ with Z/T_ϕ where $T_\phi = T \cap G_\phi^{\mathrm{ss}}$, just as in Section 3. The action by T_z is locally free, hence there is an associated connection A on it. Thus the equivariant Chern class of $\Lambda^{\max}\mathrm{nor}(V_Q, V_\Delta)|_F$ is given by

$$\exp \sum_{\tilde{\lambda} \in \Lambda_Q^\perp} -2\pi i < \lambda, t - 1/4\pi^2 dA >= \exp \sum_{\tilde{\lambda} \in \Lambda_Q^\perp} -2\pi i < \tilde{\lambda}, t - t_p - 1/4\pi^2 dA >$$

where $\exp 2\pi i (t - t_p) = s$.

If $\Delta = w\mathfrak{t}_+$, then the above is replaced by

$$\exp \sum_{\tilde{\lambda} \in \Lambda_Q^\perp} -2\pi i < \lambda, wt - wt_p - 1/4\pi^2 w dA > .$$

Let $\{\gamma\}$, $\{\beta\}$, $\{\alpha\}$ be the same as in Prop. 4.3, where X_N is replaced by M_N, then the weights to $F \subset V_Q$ in M_Q^G are given by the same expressions as in Prop. 4.4, except only those λ with $\tilde{\lambda} \in \Lambda_Q$ contribute, because those in $\Lambda_Q^\perp$ are normal to M_Q or to M_Q^G.

As for the $\det_{\mathrm{nor}(F, M_Q^G)}(1 - s^{-1}e^{-\Omega})$, similar to the expressions given in the last section, we can express it in terms of the weights $\{\gamma\}$, Λ_Q, $\{\beta\}$, $\{\alpha\}$:

$$\prod_{\lambda \in \Lambda_Q} D_0^w(ws) D_a^w(ws) D_b^w(ws)$$

where D_0, D_a, D_b are the same as in Eq. (11.4) of the last section.

There might be a non-Abelian $\mathcal{K} \supsetneq$ commuting with T_ϕ, hence it acts on Z. As was shown in Step 7 and Step 8 in the last section, the presence of a non-Abelian $\mathcal{K}$ after integrating the integrand along $\mathcal{K}/T_\phi$ leaves little trace behind, except replacing the two form $1/4\pi^2 dA$ by $1/4\pi^2 B$ which is a two form on $Z/\mathcal{K}$. Therefore we deal directly with the final expression of the denominator.

So the fixed points contribution of $F \subset M_Q^G$ is given by

(12.5)
$$\begin{aligned}
&\sum_{\Delta, Q\subset\Delta} \frac{1}{|W_Q|} \sum_{F\subset Q} \frac{1}{|I(F,M_G)|} \sum_{s\in\tau I(F,M_G)} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus \Lambda^{\max}\mathrm{nor}(V_Q, V_\Delta)|_F)}{\det_{\mathrm{nor}(F,M_Q^G)}(1-s^{-1}e^{-\Omega})} \\
&= \sum_{w\in W, Q\subset w\mathfrak{t}_+} \sum_{F\subset Q} \frac{1}{|W_Q||I(F,M_G)|} \sum_{t\in\tau I(F,M_G)} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{d} \\
&d = e^{\sum_{\lambda\in\Lambda_Q^\perp} 2\pi i<\lambda, wt-wy>} \prod_{\lambda\in\Lambda_Q} (1-e^{2\pi i<\lambda, wt-wy>}) D_0^w(ws) D_b^w(ws) D_a^w(ws),
\end{aligned}$$

where $y = t_p + 1/4\pi^2 B$.

Recall that $D_0^w(ws) = D_0(s)$ as shown in Prop. 4.4, and use the expressions from Step 2 in the last section to obtain:

$$\begin{aligned}
D_b^w(ws) &= (-1)^{\sigma(w)} e^{2\pi i \sum_{w\beta<0} 2\pi i<w\beta, y>} \prod_{\beta\in\Delta_+(\mathfrak{g}_\phi)} (1-e^{2\pi i<\beta,y>}) \\
&= (-1)^{\sigma(w)} e^{2\pi i<w\rho_\phi-\rho_\phi, y>} D_b(s); \\
D_a^w(ws) &= (-1)^{\sigma(w)} e^{-\sum_{w\alpha<0} 2\pi i<w\alpha,t>} \prod_{\beta\in\Delta_+(\mathfrak{g}_\phi)} (1-e^{-2\pi i<\alpha,t>}) \\
&= (-1)^{\sigma(w)} e^{2\pi i<\rho_\phi - w\rho_\phi, t>} D_a(s).
\end{aligned} \tag{12.6}$$

This is the same procedure as in Eq. (11.23). From Lie theory, we have

$$\sum_{\lambda\in\Lambda_Q} w\lambda + \sum_{\lambda\in\Lambda_Q^\perp} w\lambda = w\rho_\phi.$$

Now the denominator can be written as

(12.7)
$$\begin{aligned}
d =& e^{\sum_{\lambda\in\Lambda_Q^\perp} <2\pi i\lambda, wt-wy>} \prod_{\lambda\in\Lambda_Q} (1-e^{2\pi i<\lambda, wt-wy>}) D_0(\tau) D_b^w(w\tau) D_a^w(w\tau) \\
&= (-1)^{\sigma(w)+\sigma(w)} D_0(\tau) D_b(\tau) D_a(\tau) e^{2\pi i<\tilde{\rho}, t-y>} \prod_{\lambda\in\Lambda_Q} (e^{-2\pi i<\lambda, wt-wy>} - 1)).
\end{aligned}$$

The number of terms with a fixed Q, and varying Δ is exactly $|W_Q|$ which is the same number of Δ containing Q. Gather them together and get rid of the term $1/|W_Q|$. Now the summation over all the $Q \subset \Delta = w\mathfrak{t}_+$ containing $\phi(F)$ to yield the following:

(12.8)
$$\begin{aligned}
&\sum_{Q\subset\Delta} \frac{1}{|W_Q|} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus \Lambda^{\max}\mathrm{nor}(V_Q, V_\Delta)|_F)}{\det_{\mathrm{nor}(F,M_A^G)}(1-s^{-1}e^{-\Omega})} \\
&= \sum_{Q\subset\Delta} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{D_0(\tau)D_b(\tau)D_a(\tau)e^{2\pi i<\rho_\phi, t-y>})} \sum_Q \frac{1}{\prod_{\lambda\in\Lambda_Q}(e^{-2\pi i<w\lambda, t-y>} - 1)} \\
&= \sum_{Q\subset\Delta} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{D_0(\tau)D_b(\tau)D_a(\tau)e^{2\pi i<\rho_\phi, t-y>}} \sum_Q \frac{(-1)^{\#\Lambda_Q}}{\prod_{\lambda\in\Lambda_Q}(1-e^{-2\pi i<\lambda, wt-wy>})}.
\end{aligned}$$

Summing over various Δ in the above notations is the same as going over $w \in W_\phi$, thus the total contributions of F is 0 since

$$\sum_{w \in W_\phi} \sum_Q \frac{(-1)^{\# \Lambda_Q}}{\prod_{\lambda \in \Lambda_Q} (1 - e^{w\lambda})} = 0$$

by Eq. (6.7).

Thus only those $\{F\}$ on V_Q with $\phi(F)$ not on the boundary contributes. They are exactly the T-fixed point components on V. Therefore, the right hand side is the same as the left one. QED

13. Expression for the remainder term $\mathcal{R}$

If for each fixed point set component F of Y, FC_F has no pole on

$$\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}}|\lambda \in P_+^k\},$$

then the cancellation of Section 11 shows all the T-fixed points on the boundary cancels on the subset. However, such poles may occur, thus the fixed points coming from compactification do not cancel readily. The question is how they contribute. Here we will find an explicit expression for the remainder term $\mathcal{R}$.

First let's outline the steps in calculating the remainder term $\mathcal{R}$ as function on $\{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^\vee}}|\lambda \in P_+^k\}$. Instead of using the fixed points, we will construct varieties which contain the fixed points on the compactifying locus. Those varieties are obtained using cuts which are transversal to ∂wC. Instead of transporting the fixed points by translating elements in W^{aff}, which fails to work if the singular poles occur, we will move the varieties. The varieties have equivariant Riemann-Roch which are well defined functions on T.

After transporting the varieties using the appropriate elements in W_{aff}, one can cancel the contributions of fixed points lying on the compactifying locus, or on the boundary of certain Weyl chambers, just as was done in the last section. Then we will apply Prop. 12.1 and transport only $\{V_Q\}$ back to get a explicit formula of $\mathcal{R}$.

13.1. **Partition of the affine alcove.** The union $\bigcup_{w\in W^{\mathrm{aff}}} wC$ forms a tiling of $\mathfrak{t}$. Let x be in the interior of C, then $W^{\mathrm{aff}}(x)$ is the vertices of a dual W^{aff}-invariant decomposition of $\mathfrak{t}$.

Each vertex a of C is now the center of a convex polytope with $W_v^{\mathrm{aff}}(x)$ as vertices.

Let l be the rank of G, a little experiment shows that under the W^{aff} translations there are $l+1$ of different polytopes in the dual to $\bigcup_{w\in W^{\mathrm{aff}}} wC$. Each one can be moved so that it contains exactly one of the vertices of C as an interior point.

Denote those $l+1$ polytopes by $\{R_a\}$. Let $(LG)_a$ be the subgroup of LG preserving

$$(a, 1) \in \mathfrak{t} \times \{k\} = \mathfrak{t} \times \mathbb{R} \subset \widetilde{l\mathfrak{g}},$$

under the co-adjoint action of the central extension. By construction R_a is invariant under W_a^{aff} which is a subgroup of W^{aff} and is the Weyl group of $(LG)_a$. The invariance comes from the fact that the edges of R_a are spanned by affine roots, hence invariant under the reflection by the corresponding root.

Let $U_a = \mu^{-1}((LG)_a(R_a))$, then U_a is a symplectic $(LG)_a$-manifold of finite dimension, since μ is proper and $(LG)_a(R_a)$ is compact in $\widetilde{l\mathfrak{g}}$. The proof that it is symplectic is identical to the finite dimensional situation.

For $H \subset T$, the H-fixed points on X_N have images under ϕ as linear subsets in kC, denote them by $\{P_d\}$, $0 \le d \le l-1$ is the dimension of the connected component. And each P_d is a relatively open set in kC. The following is easy to verify:

Lemma 13.1. *There is a point $x \in C$, rational w.r.t. the weight lattice and in the interior or of C, such that the corresponding decomposition of $\mathfrak{t}$ as described above is transversal to all the linear subsets $\{P_d/k\}$.*

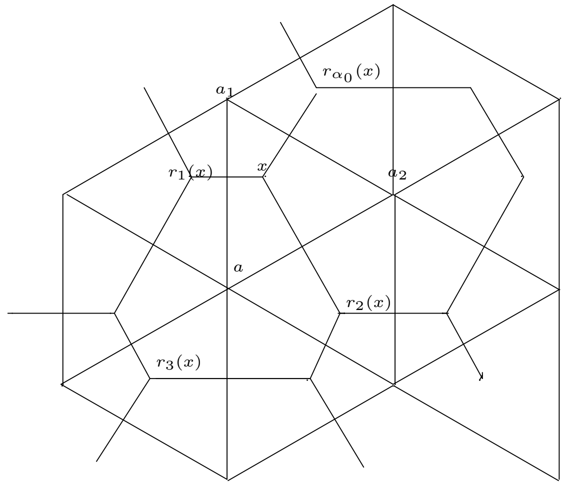

FIGURE 13.1. Partition of $\mathfrak{t}, wC$

13.2. **Varieties corresponding to the partition.** .

For each R_a, $W_a^{\text{aff}}(R_a \cap C) = R_a$. So $R_a \cap C$ acts as the fundamental domain of W_a^{aff} on R_a.

Let the faces of R_a be denoted by $\{\Box\}$, then each $\Box$ is preserved by a subgroup in W_a^{aff}, $W_\Box$. Because each face of R_a is spanned by roots, and the reflections defined by those roots preserve $\Box$. Anther way to see $W_\Box$ is as follows: If Q is the smallest face of wC which meets $\Box$, then the intersection has to be a point, otherwise even a smaller face can be found to meet $\Box$. Let $p = \Box \cap Q$, then $\Box$ is perpendicular to Q, since it is defined by the roots vanishing on Q. Therefore the group $W_Q \subset W^{\text{aff}}$ which fixes Q, preserves $\Box$.

Definition 13.1. *Let $(LG)_\Box$ denote the group which fixes the smallest face Q intersecting $\Box$. $W_\Box \subset W^{\text{aff}}$ be the subgroup preserving $\Box$.*

Next we shall use $\{\Box\}$ to define symplectic cuts on both $X, Y = G \times_T X_N$.

For each face $\Box$, the intersection $\Box \cap C$ is a convex polytope. The map $\phi : X_N \to kC$ defines for each $\Box$, $\phi^{-1}(k(C \cap \Box))$. For simplicity of notations, let $k = 1$.

For each sub-face B of $\Box$ in the interior of C, let $\mathfrak{t}_B$ be the subalgebra of $\mathfrak{t}$ perpendicular to the linear set defined by the face B, T_B be the group generated by $\mathfrak{t}_B$.

Definition 13.2. *Let $X_{N,\Box}$ be the cut space associated with $\phi^{-1}(C \cap \Box)$, i.e.,*

$$X_{N,\Box} = \cup_B \phi^{-1}(B)/ \simeq .$$

That $X_{N,\Box}$ is an T-orbifold with an invariant almost complex structure follows from the same argument for X_N itself. Although $X_{N,\Box}$ has degenerate symplectic form, just as X_N does. The orbifold line bundle L_N also induces one on $X_{N,\Box}$, denoted by $L_{N,\Box}$ which can be defined using quotients by T_B on $L_N|\phi^{-1}(C \cap \Box)$.

The proof of the following can be found in [M]:

Lemma 13.2. *The T-equivariant Riemann-Roch satisfies the following*

$$\mathrm{RR}_T(X_N, L_N) = \sum_{\Box \cap C \neq \emptyset} (-1)^{\operatorname{codim} \Box} \mathrm{RR}_T(X_{N,\Box}, L_{N,\Box}).$$

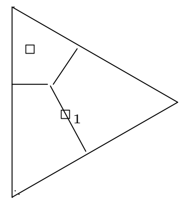

FIGURE 13.2. The ranges of the images of $X_{N,\square}$

Therefore the corresponding G-spaces Y, $Y_\square = G \times_T X_{N,\square}$ and the line bundles $L_Y, L_\square$ satisfy

$$\mathrm{RR}_G(Y, L) = \sum_{\square \cap C \neq \emptyset} (-1)^{\mathrm{codim}\square} \mathrm{RR}_G(Y_\square, L_\square). \tag{13.1}$$

Remark: Readers should compare this surgery formula with that in Prop. 12.1.

There is a map $\phi : Y_{G,\square}$ which serves as moment map, for the degenerate symplectic form, just as we have shown for $Y = G \times_T X_N$.

The image $\phi(Y_{G,\square}) \cap \mathfrak{t}$ is contained in $W(k(\square \cap C))$, since the image of $X_{N,\square}$ is in $(k(\square \cap C))$.

13.3. **Another space $Z_\square$ associated with $\square$.** Define $Z_\square$ to be the $(LG)_\square$-symplectic orbifold which is the cut space associated with $\mu^{-1}(LG_\square(\square))$. There are two ways of defining it. The first one is as in [C3], where a holomorphic $LG \times (LG)_\square$ symplectic orbifold $\mathcal{M}_\square$ was constructed, then $Z_\square$ can be defined as the symplectic reduced space of the product $X \times \mathcal{M}_\square$.

The other approach is given in [M].

The space $Z_\square$ is compact, has a moment map Φ whose image $\Phi(Z_\square \cap \mathfrak{t}) \subset \square$.

If $LG_\square \subseteq G$, i.e., $\square \cap C^{\mathrm{aff}} = \emptyset$, the two space $G_\square \times_T X_{N,\square}, Z_\square$ are related as twins in the sense of the previous section, where M, M_G would be $Z_\square$ and $G_\square \times_T X_{N,\square}$ respectively. In particular they share the same Riemann-Roch.

On the other hand if $\square \cap C^{\mathrm{aff}} \neq \emptyset$, then we will see below how the Riemann-Roch are related.

13.4. **Riemann-Roch of $X_{G,\square}$ and $Z_\square$.** For $w \in W$, clearly $w(\square \cap C) \subset wC$. If $W^{\mathrm{aff}}_\square$ preserves $\square$, then $Ad_w W^{\mathrm{aff}}_\square$ preserves $w\square$, $w \in W$.

In the following $W_\square$ will be used in place of $W^{\mathrm{aff}}_\square$.

For $W_\square \subset W^{\mathrm{aff}}$, there is the isomorphic subgroup $W^0_\square \subset W$. The proof of this simple fact is identical to that of the last statement in Prop. 7.2.

For each w is in $W^0_\square$, let w' denote the corresponding element in $W_\square$.

Lemma 13.3. *For every pair w, w', there is a translating element v in the long root lattice so that $w = vw'$.*

Pf: This is a fairly simple fact. Since I can not find a reference for that, the proof is included. The proof is based on induction of the length of w.

By applying an element in W, we may assume that $\square \cap C \neq \emptyset$ since W preserves the long root lattice. For such a $\square$, all the simple roots of $(lg)_\square$ are all simple, or contain α_0.

Suppose w is r_i, then

$$r_i r_i' = r_i (r_i')^{-1}$$

is either I, or $-\theta$ depending on whether r_i' is defined by a simple root of $\mathfrak{g}$ or by α_0. Thus the assertion holds for elements of length 1.

Assume it's proved for elements with length less n. Suppose w has length n, $w = rw_1$ with w_1 of length less n, and r is one of the generating reflections. By induction assumption $w_1 = v_1 w_1'$, and v_1 is a translation. One can write $w' = r'w_1'$, then

$$w = rv_1w_1' = rv_1r^{-1}rr'w_1' = rv_1r^{-1}rr'w' = vw',$$

where both rv_1r^{-1}, rr' are translations by elements in the long root lattice. So is v as the composition of the two. QED

The following is essential, it explains in geometric context the appearance of the lattice $\frac{M^*}{k+h^{\rm v}}$. By the definition of M^*, we have $e^{(k+h)v} = 1 \quad \text{on} \quad \frac{M^*}{k+h^{\rm v}}$. The first part generalizes Prop. 7.2.

Proposition 13.1. *1). Let $\rho_\square, \rho$ be the half sum of positive roots of $(lg)_\square, \mathfrak{g}$ respectively. Let $D_\square, D$ be their Weyl denominators, and $w = vw'$ with w, w', v as in the previous lemma.*

Suppose λ is of level k, then

$$w\frac{e^\lambda}{D} = e^{(k+h^{\rm v})v}\frac{D_\square}{D}w'\frac{e^\lambda}{D_\square}. \tag{13.2}$$

In particular,

$$w\frac{e^\lambda}{D} = \frac{D_\square}{D}w'\frac{e^\lambda}{D_\square} \quad \text{on} \quad \frac{M^*}{k+h^{\rm v}}. \tag{13.3}$$

where v is the translation in the previous lemma.

2). As function on T, the following holds

$$w\int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus H)}{\det(1-t^{-1}e^{-\Omega})} = e^{(k+h^{\rm v})v}\frac{D_\square}{D}w'\int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus H)}{\det(1-t^{-1}e^{-\Omega})} \tag{13.4}$$

where H is a bundle of level 0 on which $W^{\rm aff}$ acts. If there is no pole on $\frac{M^*}{k+h^{\rm v}}$, then

$$w\int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus H)}{\det(1-t^{-1}e^{-\Omega})} = \frac{D_\square}{D}w'\int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus H)}{\det(1-t^{-1}e^{-\Omega})} \quad \text{on} \quad \frac{M^*}{k+h^{\rm v}}. \tag{13.5}$$

Pf: By conjugation, we may assume without loss of generality that $\square \cap C \neq \emptyset$.

The following is equivalent to the assertion, after applying the well-known transformation rule on the Weyl denominator:

$$e^{w\lambda+\rho-w\rho} = e^{(k+h^{\rm v})v}e^{w'\lambda+\rho_\square-w'\rho_\square}$$

which can be verified the same way as Prop. 7.2, replacing ρ_μ there by $\rho_\square$.

To see the second part, lift w, w to $N(T)$, they act on F, the normal bundle and L_F. Notice that the line bundle has level k, by assumption on the moment map μ, i.e. the central part of $\widetilde{l\mathfrak{g}}$ acts with weight k on the fiber. But the central part acts trivially on X, the action on the normal bundle is of weight 0. Thus if λ is a weight on the normal bundle, we have $w\lambda = w'\lambda$. Hence, we have the desired identity as a consequence of part 1). QED

13.5. **Moving $X_{N,\Box}$ and the consequence.** As we have mentioned earlier when FC_F has poles on $\frac{M^*}{k+h^{\mathrm{v}}}$, we can not replace $w\mathrm{FC}_F$ by $\frac{D_\Box}{D}w'\mathrm{FC}_F$ on $\frac{M^*}{k+h^{\mathrm{v}}}$. On the other hand, the function $\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})$ is a polynomial, thus it can be evaluated everywhere. For that function, the following holds:

The above transformation rule yields the following

Corollary 13.1.

$$w\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D} = e^{(k+h^{\mathrm{v}})v}\frac{D_\Box}{D}w'\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D_\Box}; \tag{13.6}$$

Furthermore,

$$w\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D} = \frac{D_\Box}{D}w'\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D_\Box} \quad \text{on} \quad \frac{M^*}{k+h^{\mathrm{v}}} \tag{13.7}$$

For $X_{G,\Box} = G \times_T X_{N,\Box}$, one has

$$\mathrm{RR}(X_{G,\Box}, L_{G,\Box}) = \sum_{u\in W/W^0_\Box} u\frac{D_\Box}{D}\cdot u\mathrm{RR}(Z_\Box, L_\Box) \quad \text{on} \quad \frac{M^*}{k+h^{\mathrm{v}}}. \tag{13.8}$$

Pf: The first one is an immediate consequence of Prop.13.1, after one writes both sides in terms of the T-fixed points contributions.

Since the function $\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})$ is the equivariant index of a *spin*$\mathbb{C}$ complex defined by the pair $X_{N,\Box}, L_{N,\Box}$, it is a well defined function everywhere. Hence we can evaluate on the lattice $\frac{M^*}{k+h^{\mathrm{v}}}$ to get the second formula.

To see the next identity, expand $\mathrm{RR}(X_{G,\Box}, L_{G,\Box})$ in terms of $\mathrm{RR}(X_{N,\Box}, LN, \Box)$, then apply the first and second identity to yield

(13.9)

$$\begin{aligned}\mathrm{RR}(X_{G,\Box}, L_{G,\Box}) &= \sum_{w\in W} w\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D} \\ &= \sum_{u\in W/W^0_\Box, w\in W^0_\Box} uw\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D} \\ &= \sum_{u\in W/W^0_\Box} u\Big(\frac{D_\Box}{D}\sum_{w'\in W_\Box} w'\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D_\Box}\Big) \quad \text{on} \quad \frac{M^*}{k+h^{\mathrm{v}}},\end{aligned}$$

it is easy to recognize the sum $\sum_{w'\in W_\Box} w'\frac{\mathrm{RR}_T(X_{N,\Box}, L_{N,\Box})}{D_\Box}$ is simply the Riemann-Roch of the space $(LG)_\Box \times_T X_{N,\Box}$. How is it related to $Z_\Box$? They are twin-pairs as discussed in the last section, replacing M, G there by $Z_\Box, (LG)_\Box$. Thus one has

$$\mathrm{RR}(X_{G,\Box}, L_{G,\Box}) = \sum_{u\in W/W^0_\Box} u\Big(\frac{D_\Box}{D}\mathrm{RR}(Z_\Box, L_\Box)\Big) \quad \text{on} \quad \frac{M^*}{k+h^{\mathrm{v}}}. \quad \text{QED} \tag{13.10}$$

So the above relates the Riemann-Roch of G-space $X_{G,\Box}$ and the $LG_{u\Box}$-space $\{Z_{u\Box} := uZ_\Box\}$.

The fig 13.5 illustrates the relations between the intersections with $\mathfrak{t}$ of the images of $X_{G,\Box}$ and three $Z_{u\Box}$. The three separate regions, inside the middle hexagon, with dotted lines are associated with $X_{G,\Box}$. In this case $W^0_\Box \simeq W$, the image of $Z_\Box$ meeting $\mathfrak{t}$ inside one of the regions filled with dashed lines. What are the other two identical regions? If one starts with $u\Box$, in this case $u \neq I, r_3$, then $W^0_{u\Box} \simeq W$ holds as well. And one of the other two regions will contain the intersection of the image with $\mathfrak{t}$ of $uZ_\Box$.

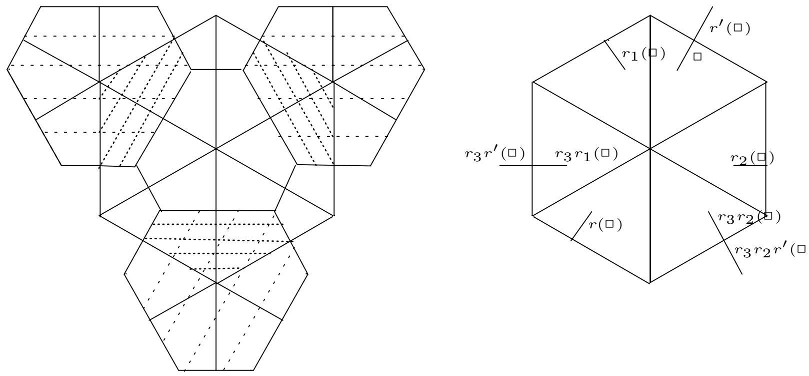

FIGURE 13.3. Locations of the images of $X_{G,\square}$ and $\{Z_{u\square}\}$.

In the second figure, $W_\square \simeq \mathbb{Z}_2$, thus $W^0_\square$ is not the same as W, $W/W^0_\square$ has three elements. The union of the six short segments inside the hexagon contains $\mu(X_{G,\square}) \cap \mathfrak{t}$. The long segments contain the intersections with $\mathfrak{t}$ of the images of $\{uZ_\square\}$.

13.6. **The final step.** *Proof of the Main Theorem: Step 1:* By cutting and applying Cor. 13.1, we first cancel the contributions to $\mathrm{RR}_G(Y)$, from the fixed point set F with $\phi(F) \in w(\partial C \setminus C^{\mathrm{aff}})$. The cancellation below is implicit when applying Cor. 13.1.

By the fundamental property of cutting, one has

$$\begin{aligned}\mathrm{RR}(Y) &= \sum_{w\in W} w\frac{\mathrm{RR}(X_N)}{D}\\ &= \sum_{w\in W} w\frac{1}{D}\sum_{\square\cap C\neq\emptyset}(-1)^{\mathrm{codim}\square}\mathrm{RR}_T(X_{N,\square})\\ &= \sum_{\square\cap C\neq\emptyset}(-1)^{\mathrm{codim}\square}\sum_{w\in W} w\frac{1}{D}\mathrm{RR}_T(X_{N,\square})\\ &= \sum_{\square\cap C\neq\emptyset}(-1)^{\mathrm{codim}\square}\sum_{u\in W/W^0_\square} u\frac{D_\square}{D}u\mathrm{RR}(Z_\square) \quad \text{on} \quad \frac{M^*}{k+h^{\mathrm{v}}}\end{aligned}$$

where the last step uses Cor. 13.1. See the figure for the translations of the images of $\phi(X_{N,\square})$, similar actions are taken place for the varieties with images lying on the lower dimensional $\square$ which are not illustrated.

Step 2: Localize to V. It should be clear that $u\mathrm{RR}(Z_\square) = \mathrm{RR}(Z_{u\square})$. While the intersection of the image of $Z_\square$ with $\mathfrak{t}$ is in $W^{\mathrm{aff}}_\square(\square)$, the image of $uZ_\square$ is in

$$uW^{\mathrm{aff}}_\square(\square) = \mathrm{Ad}_u(W^{\mathrm{aff}}_\square)(u\square) = W^{\mathrm{aff}}_{u\square}(u\square).$$

Each $\tau \in \{e^{2\pi i\nu^{-1}\frac{\lambda+\rho}{k+h^{\mathrm{v}}}}|\lambda \in P^k_+\}$ is a generic element in T, hence the connected components of τ-fixed points in $Z_{u\square}$, denoted by $\{V\}$, must have image under moment map in $\mathfrak{t}$. Each V is an orbifold, and is symplectic, by general theory on fixed point sets and the fact that $Z_{u\square}$ is a symplectic orbifold. We may apply Prop. 12.1, replacing M, G, V there by $Z_{u\square}, K, V$ where $K = (LG)_{u\square}$. The collection $\{\Delta\}$

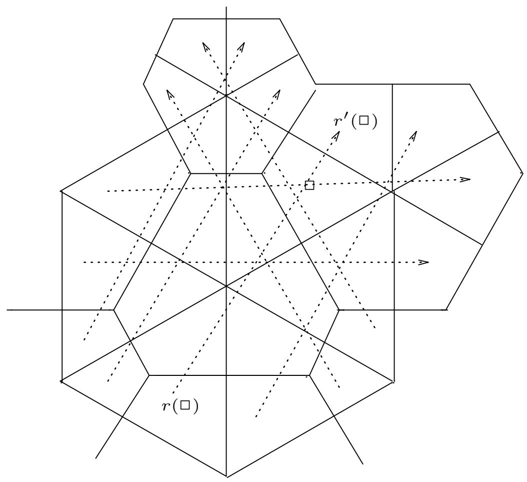

FIGURE 13.4. Comparing varieties with images differed by translation elements in W^{aff}

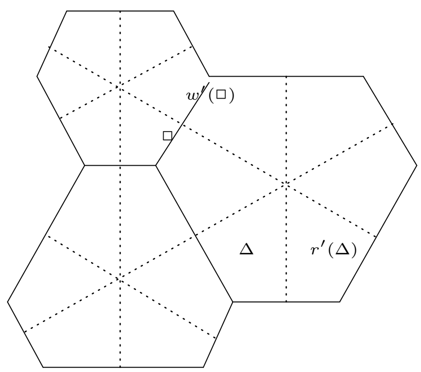

FIGURE 13.5. After translation and the twin pair comparison

there is now replaced by $\{uw'(\Box \cap C) | w' \in W^{\mathrm{aff}}_{\Box}\}$ which is denoted by $\{\Delta'\}$ in the following. Denote by Q_1 a face of $u(\Box \cap \partial C)$, and

$$Q = \mathrm{Ad}_u\, w(Q_1), \quad Q' = \mathrm{Ad}_u\, w'(Q_1)$$

where $w' \in W^{\mathrm{aff}}_{\Box}$ and w is the corresponding element in $W^0_{\Box}$. We obtain

$$\begin{aligned}
& u\big(\mathrm{RR}(Z_\Box)\big)(\tau) \\
&= \mathrm{RR}(Z_{u\Box})(\tau) \\
&= \sum_{\Delta'} \sum_{Q' \subset \Delta'} \frac{1}{|W_{Q'}||I_{V_{Q'}}|} \sum_{t \in \tau I_{V_{Q'}}} \int_{V_{Q'}} \frac{\mathrm{Td}(V_{Q'})\mathrm{Ch}(L_{V_{Q'}} \oplus \Lambda^{\max}\mathrm{nor}(V_{Q'}, V_{\Delta'}))}{\det_{\mathrm{nor}(V_{Q'}, X^K_{Q'})}(1 - t^{-1}e^{-\Omega})}.
\end{aligned}$$

The varieties $\{Z_{u'\Box}\}$ have images as in the following figure:

Step 3: Transporting $V_{Q'}$. The transporting here are in the directions opposite to the arrows in the figure 13.6. One does not simply get back the result in the

beginning, since the cancellation has already taken place. This is an important realization.

Two observations can be made here: Each integral $\int_{V_{Q'}}$ in the above is finite, when evaluated at τ, because $V_{\mathrm{Ad}_u w'(Q)}$ is a connected component of τ-fixed points in Z_Q^K, so the action by τ on the normal bundle is non-trivial, thus the denominator is well defined at τ. Second observation is that one can apply Prop. 13.1 2) to this situation. To make this more explicit, let $\mathrm{Ad}_u w = s\,\mathrm{Ad}_u w'$ where s is a translation defined by an element in the long root lattice, according to Lemma 13.3. For each $\mathrm{Ad}_u w'(\square\cap C)$, there is the corresponding $\mathrm{Ad}_u w(\square\cap C)$, obtained by shifting $-s$. Likewise $\mathrm{Ad}_u w(Q_1)$ is obtained from $\mathrm{Ad}_u w'(Q_1)$ by shifting $-s$. The corresponding varieties $V_Q\subset X_Q$ have a similar relation $s^{-1}(V_Q)=V_{-s(Q)}\subset s^{-1}(X_Q)=X_{-s(Q)}$, and $s^{-1}(V_Q)$ is a connected component of τ-fixed points, since the translation commutes with T-action.

The denominator is given by

$$\det_{\mathrm{nor}(V_{Q'},X^K_{Q'})}(1-t^{-1}e^{-\Omega}) = uw'D_\square \det_{\mathrm{nor}(V_Q,X_Q)}(1-t^{-1}e^{-\Omega}),$$

thus we obtain the following relation as a consequence of Prop. 13.1:

$$\begin{aligned}&e^{(k+h^{\vee})s}u\frac{D_\square}{D}\int_{V_{Q'}}\frac{\mathrm{Td}(V_{Q'})\mathrm{Ch}(L_{V_{Q'}}\oplus\Lambda^{\max}\mathrm{nor}(V_{Q'},V_{\Delta'}))}{\det_{\mathrm{nor}(V_{Q'},X^K_{Q'})}(1-t^{-1}e^{-\Omega})}\\ &=\int_{V_Q}\frac{\mathrm{Td}(V_Q)\mathrm{Ch}(L_{V_Q}\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta))}{\det_{\mathrm{nor}(V_Q,X^G_Q)}(1-t^{-1}e^{-\Omega})}\end{aligned}$$

where $Q'=uw'(Q_1), Q=uw(Q_1)$ with Q_1 a face of $\square\cap C$.

Both sides have no poles at τ since $t\in\tau I_{V_{Q'}}$ acts on the normal bundles with no 0 eigenvalue. Evaluate them at $\tau\in\exp(\frac{M^*}{k+h^{\vee}})$, i.e. $e^{(k+h^{\vee})s}=1$, for $Q=sQ'$, $\Delta=s\Delta'$, one has the following

$$\begin{aligned}&\mathrm{RR}(Y)(\tau)\\ &=\sum_{\square\cap C\neq\emptyset}(-1)^{\mathrm{codim}\square}\sum_{u\in W/W^0_\square}u\sum_{w\in\mathrm{Ad}_u W_\square}\frac{1}{|W_Q||I_{V_Q}|}\sum_{t\in\tau I_{V_Q}}\mathcal{I}_{V_Q},\\ &\mathcal{I}_{V_Q}=\int_{V_Q}\frac{\mathrm{Td}(V_Q)\mathrm{Ch}(L_{V_Q}\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta))}{\det_{\mathrm{nor}(V_Q,X^G_Q)}(1-t^{-1}e^{-\Omega})}.\end{aligned}\tag{13.11}$$

Step 4: Eliminate the extra things. Next, we will write the above in a concise form. To see further cancellation, it is easiest to write the integrals above in terms of the T-fixed points contribution:

$$\begin{aligned}&\frac{1}{|I_{V_Q}|}\sum_{t\in\tau I_{V_Q}}\int_{V_Q}\frac{\mathrm{Td}(V_Q)\mathrm{Ch}(L_{V_Q}\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta))}{\det_{\mathrm{nor}(V_Q,X^G_Q)}(1-t^{-1}e^{-\Omega})}\\ &=\sum_{F\subset V_Q}\frac{1}{|I_F|}\sum_{t\in\tau I_F}\int_F\frac{\mathrm{Td}(F)\mathrm{Ch}(L_F\oplus\Lambda^{\max}\mathrm{nor}(V_Q,V_\Delta|_F))}{\det_{\mathrm{nor}(F,X^G_Q)}(1-t^{-1}e^{-\Omega})}.\end{aligned}$$

Substitute the above into Eq. (13.11), then we will treat those terms according to the type of F:

a). $\phi(F)$ is on a cut defined by $\square_1$ with $\dim\square_1<l=\dim\mathfrak{t}$.

b). $\phi(F)$ is on some $wQ, Q\subset\partial C\setminus C^{\mathrm{aff}}$,; but not on $\square_1$ with $\dim\square_1<l=\dim\mathfrak{t}$.

c). $\phi(F)$ is on some wQ with $Q\subset C^{\mathrm{aff}}$; but not on $\square_1$ with $\dim\square_1<l=\dim\mathfrak{t}$.

d). $\phi(F)$ is neither on any $wQ, \forall w \in W, \forall Q \subset \partial C$ nor on $\Box_1$ with $\dim \Box_1 < l = \dim \mathfrak{t}$.

Obviously, the above covers all possibilities for $\phi(F)$. We claim the contributions of F of the first two kinds amount to 0 in Eq. (13.6).

In the first case, suppose $u(\Box_1 \cap C), u(Q_1)$ are the smallest among $u(\Box \cap C), u(Q)$ containing $\phi(F)$. Fixing a $u(Q) \supseteq u(Q_1)$, and $u(\Box) \supseteq u(\Box_1)$, what is the integrand in the above integral?

For convenience of notations, assume below that $u = I$.

Let us take a moment to discuss the weights on $\mathrm{nor}(F, V_{Q\cap\Box})$ and the bundle $\mathrm{nor}(V_{Q\cap\Box}, V_\Box)|_F$. Suppose that $\mathbf{A}_\Box = \{\tilde{a}\}$ is a basis of weights in $\Box$ normal to $\Box_1$. By the construction of the cut space we know in $\mu^{-1}(\mathfrak{t})$, there is the set $\tilde{F}$ whose quotient under the subgroups $T_Q \times T_{\Box_1}$ is F, here $\mathrm{Lie}T_Q$ is normal to Q, $\mathrm{Lie}T_{\Box_1}$ is normal to $\Box_1$. Now assume that T_P stabilizes a generic point on $\tilde{F}$, then F is fixed by T iff

$$\mathrm{Lie}T_P \oplus \mathrm{Lie}T_{\Box_1} \oplus \mathrm{Lie}T_Q = \mathfrak{t}.$$

The proof of the above follows from that of Lemma 3.1. Let $\{\tilde{\lambda}\}, \{\tilde{\gamma}\}$ be weights of $\mathfrak{t}_Q, \mathfrak{t}_P$ respectively, as in Prop. 4.3, (replacing $\mathfrak{t}_z$ there by $\mathfrak{t}_Q$), and let $\tilde{\beta}$ be the roots of the group $(LG)_Q$ which is the stabilizer in LG of Q.

Each $t \in T$ comes from a product of a triple

$$t_P \cdot t_{\Box_1} \cdot t_Q \in T_P \cdot T_{\Box_1} \cdot T_Q,$$

the finite ambiguity of the choice of the triple causes F to be an orbifold singularity. So the orbifold weights on the normal bundle of F in $X_{Q\cap\Box}$ can be described as follows:

$$\begin{aligned} &\lambda(t) = \tilde{\lambda}(t_Q); \quad \gamma(t) = \tilde{\gamma}(t_P); \\ &a(t) = \tilde{a}(t_{\Box_1}); \quad \beta(t) = \tilde{\beta}(t_P) \end{aligned} \tag{13.12}$$

the verification of the above is the same as Prop. 4.3.

The normal bundle $\mathrm{nor}(V_{Q\cap\Box}, V_{C\cap\Box})|_F$ has weights given by $\lambda(t) = \tilde{\lambda}(t_Q), \tilde{\lambda} \notin \Lambda_Q$ where Λ_Q is the subset of $\{\tilde{\lambda}\}$ not parallel to Q.

One realizes in the above that for different $\Box \supseteq \Box_1$, the only difference in the integrand is the term $\prod_{a\in\mathbf{A}_\Box}(1 - e^{-2\pi i<a,t+1/4\pi^2 dA>})$ appearing in the denominator, here B is the form representing the Chern class of the principle bundle corresponding to the orbit of $T_{\Box_1}$.

Let $D_a(s), D_b(s)$ be the same as in Section 11, and D_0' be defined the same as D_0 except only those γ tangent to $\Box_1$ will be involved. We have the following representation of the integrand:

$$\begin{aligned} &(-1)^{\mathrm{codim}\Box} \sum_{\Box\supseteq\Box_1} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F \oplus \Lambda^{\max}\mathrm{nor}(V_{Q\cap\Box}, V_{C\cap\Box})|_F)}{\det_{\mathrm{nor}(F, X^G_{Q\cap\Box})}(1 - s^{-1}e^{-\Omega})} \\ &= \sum_{\Box\supseteq\Box_1} \int_F \frac{\mathrm{Td}(F)\mathrm{Ch}(L_F)}{e^{\sum_{\tilde{\lambda}\notin\Lambda_Q} 2\pi i<\lambda, t-t_P-1/4\pi^2 dA>} D_a(s)D_b(s)D_0'(s)} \\ &\quad \times \sum_{\mathbf{A}_\Box} (-1)^{\mathrm{codim}\Box} \frac{1}{\prod_{a\in\mathbf{A}_\Box}(1 - e^{-<a,t_P+1/4\pi^2 dA>})} \\ &= 0 \end{aligned} \tag{13.13}$$

since

$$\sum_{\mathbf{A}_\square}(-1)^{\operatorname{codim}\square}\frac{1}{\prod_{a\in\mathbf{A}_\square}(1-e^{-<a,t_P+1/4\pi^2 dA>})}=0.$$

As for F of the second type, $\phi(F)$ is in the interior of $\cup_w w(C)$, but on a wall of Weyl chamber $w(Q)$. The sum of the integrals over various Q' with $Q\subset Q'$ was already shown to be 0 in the proof of the Prop. 12.1. The argument there for those fixed points appearing on the boundary of the Weyl chamber shows the same cancellation here.

Therefore, only F in c) or d) survives. We do recognize these two types, F of type c) are those produced by the intersection of τ fixed point set V, with the compactification locus corresponding to $w(C^{\mathrm{aff}})$, i.e. the affine walls. Those of type d) are the fixed point sets in X with images in $W(C^{\mathrm{int}})$. Thus we have the following form for $\mathrm{RR}(Y)$ when evaluated at τ:

(13.14)

$$\begin{aligned}\mathrm{RR}(Y) = &\sum_{\{F|\phi(F)\in W(C^{\mathrm{int}})\}}\mathrm{FC}(F)\\ &+\sum_{\{F|\phi(F)\in W(C^{\mathrm{aff}})\}}\frac{1}{|W_Q||I_F|}\sum_{t\in\tau I_F}\int_F\frac{\mathrm{Td}(F)\mathrm{Ch}\,(L_F\oplus\Lambda^{\max}\mathrm{nor}(Y_Q,Y)|_F)}{\det_{\mathrm{nor}(F,Y_Q)}(1-s^{-1}e^{-\Omega})}.\end{aligned}$$

The first sum is over the true fixed points on $X\cap\mu^{-1}(W(C))$, the absence of the isotropy group I_F is due to the smoothness of Y at the interior fixed points, since no cutting passes such F. The second sum is over those fixed points lying on the intersection of the compactification locus Y_Q and a τ-fixed point set component. The only explanation needed here is that

$$\mathrm{Ch}(\mathrm{nor}(Y_Q,Y_\Delta)|_F)=\mathrm{Ch}(\mathrm{nor}(V_{Q\cap\square_1},V_{C\cap\square})|_F)$$

which is obvious in terms of these weights λ with $\tilde\lambda\in Q$.

Thus we obtain the second expression for $\mathrm{RR}(Y)(\tau)$ in the main theorem. The first one can be obtained easily now by localize the integrals over V_Δ, V_Q to their fixed points. QED.

We emphasize that there are plenty of T-fixed point sets on the compactified locus, they do not contribute to the formula in Eq. (13.14) unless they are on the compactified τ-fixed points.

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